Algebraic generalization of braid groups

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1. Artin-Tits Groups and their parabolic subgroups.
2. The parabolics of Garside structure.
3. Hecke Algebra of Renner monoids.
4. Free group automorphisms and braid representation.

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Let $M = (m_{s,t})_{s,t \in S}$ be a Coxeter matrix.

$$A = \left\langle S \mid \underbrace{sts\ldots}_{m_{s,t}} = \underbrace{tst\ldots}_{m_{s,t}} \quad s \neq t \right\rangle$$

The group A is called the Artin-Tits group associated with M.

$\rightsquigarrow$ The submonoid A^+ generated by S as the same presentation.

$\rightsquigarrow$ The associated Coxeter group is $W = A/_{s^2=1}$.

If $S = \{s_1, \cdots, s_n\}$ with $m_{s_i,s_j} = 3$ for $|i-j| = 1$ and $m_{s_i,s_j} = 2$ for $|i-j| > 1$, the associated Artin-Tits group is the braid group B_{n+1} on $n+1$ strands.
$W = \mathfrak{S}_{n+1}$.

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1}\sigma_n) \cdots (\sigma_1\sigma_2)\sigma_1$$

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Definition

(i) A monoid is said to be a locally Garside monoid if

(a) it is cancellative and Noetherian ;

(b) any two elements have a common multiple for left-divisibility if and only if they have a least common multiple for left-divisibility ;

(c) any two elements have a common multiple for right-divisibility if and only if they have a least common multiple for right-divisibility.

(ii) A Garside element of a locally Garside monoid is a balanced element whose set of factors generates the whole monoid. When such an element exists, we say that the monoid is a Garside monoid.

(iii) A (locally) Garside group $G(M)$ is the enveloping group of a (locally) Garside monoid M.

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Abstract definition

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parabolic subgroups of Artin-Tits groups

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$\rightsquigarrow$ the associated Garside monoid is A^+.

Theorem

Let $T \subseteq S$ and $N = (m_{s,t})_{s,t\in T}$

$$A_T := \langle T \rangle_A \simeq \left\langle T \mid \underbrace{sts\ldots}_{m_{s,t}} = \underbrace{tst\ldots}_{m_{s,t}} \quad s \neq t \right\rangle.$$

The subgroup A_T is called a standard parabolic subgroup of A.

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- $\rightsquigarrow$ A_T is an Artin-Tits group ;
- $\rightsquigarrow$ The Garside structures of A and A_T are compatibles ;
- $\rightsquigarrow$ $A^+ \cap A_T = A_T^+$;
- $\rightsquigarrow$ $A_T \cap A_U = A_{T \cap U}$;
- $\rightsquigarrow$ The family of parabolic subgroups allows to solve the word problem in A (for some cases).
- $\rightsquigarrow$ The family of parabolic subgroups allows to build a finite dimensional CAT(0) complex on which A acts (for some cases).
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Question : Is there a natural notion of parabolic subgroup in the framework of Garside groups ? Can we define a standard parabolic subgroup as any subgroup generated by a subset of atoms ?

Question : Is there a natural notion of Coxeter-like associated group in the framework of Garside groups ?

Remark : General Artin-Tits groups are badly understood (not even a solution to the word problem). So, one may wonder :

Why is it interesting to consider locally Garside groups which are more general ?

$\rightsquigarrow$ A better abstract understanding can help to understand Artin-Tits groups.
$\rightsquigarrow$ Natural objects of study can be equiped with a Garside structure.
$\rightsquigarrow$ Garside groups are well-understood, as spherical-type Artin-Tits groups.
$\rightsquigarrow$ Some Artin-Tits groups are (locally) Garside groups and can be understood in this way (word problem...).

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⇝ First step : What about Garside groups ?

Example

Consider $M = \langle a_1, a_2, a_3, a_4 \mid a_1a_2 = a_2a_3 = a_3a_4 = a_4a_1 \rangle$. The subgroup $G_{1,3}$ is a free group with base $\{a_1, a_3\}$, and $G_{1,3} \cap M$ is the free monoid with base $\{a_1, a_3\}$.

$$G_{1,3} \cap G_{2,4} = \langle a_3^{-1} a_1 \rangle.$$

Lemma

If M is an Artin-Tits monoid of spherical type, then its classical parabolic submonoids are in one-to-one correspondance with the balanced elements that belongs to $Div(\Delta)$.

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Definition (G)

Let M be a Garside monoid and Δ be a Garside element. If $\delta \in \mathrm{Div}(\Delta)$ balanced, then M_δ is a parabolic submonoid with δ as a Garside element if and only if

$$\mathrm{Div}(\delta) = \mathrm{Div}(\Delta) \cap M_\delta.$$

Definition (G-Paris)

Let M be a monoid and N be a submonoid. We say that N is special when it is closed by factors, that is $abc \in N \implies a, b, c \in N$.

Lemma

Let M be a locally Garside monoid and N be a special submonoid. Then N is a locally Garside monoid, and a lower sub-semilattice.

Definition (G)

Let M be a Garside monoid and Δ be a Garside element. If $\delta \in \mathrm{Div}(\Delta)$ balanced, then M_δ is a parabolic submonoid with δ as a Garside element if and only if

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Garside groups

parabolic subgroups

Example

Consider $M = \langle a_1, a_2, a_3, a_4 \mid a_1a_2 = a_2a_3 = a_3a_4 = a_4a_1 \rangle$. The submonoid $G_{1,3} \cap M$ is special submonoid of M but $G_{1,3} \cap G_{2,4} = \langle a_3^{-1}a_1 \rangle$.

Definition (G-Paris)

Let M be a Garside monoid and $G(M)$ be its associated Garside group.

(i) A submonoid of M is said to be parabolic if it is special, and closed by left lcm and by right lcm. A parabolic submonoid is of spherical type if it has a Garside element.

(ii) A subgroup of $G(M)$ is standard parabolic if it is generated by the image $\iota(N)$ of a parabolic submonoid N of M.

(iii) A subgroup is parabolic if it is conjugate to a standard parabolic subgroup.

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Example

If M is an Artin-Tits monoid, then its classical parabolic submonoids are the parabolic submonoids defined by the associated locally Garside structure.

Theorem (G-Paris)

Let M be a Garside monoid.

Every parabolic submonoid N of M is of spherical type. Moreover, there exists a Garside element Δ_N of N such that

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Conjecture (G-Paris)

For every locally Garside monoid, Properties (P1), (P2), (P3) and (P4) hold.

(P1) *The canonical morphism $\iota : M \to G(M)$ is into.*

(P2) *The group $G(M)$ is torsion free.*

(P3) *If N is a parabolic submonoid, then the associated standard parabolic subgroup is isomorphic to $G(N)$. Moreover, $G(N) \cap M = N$ in $G(M)$.*

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Definition (G-Paris)

The family of locally Garside groups of FC type is the smallest family of locally Garside groups that contains Garside groups and that is closed by amalgamation above a parabolic subgroup.

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Every locally Garside group of FC type verifies properties (P1)—(P4).

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Garside groups
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Garside groups

Application 1 : FC type Garside groups

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Consider $M =$

$$\left(\begin{array}{ccccccc|c}
1 & 3 & 2 & \cdots & & 2 & 3 & 3 \\
3 & 1 & 3 & 2 & \cdots & & 2 & 2 \\
2 & 3 & 1 & 3 & 2 & \cdots & 2 & \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 2 & \vdots \\
2 & \cdots & 2 & 3 & 1 & 3 & 2 & 2 \\
2 & & \cdots & 2 & 3 & 1 & 3 & 3 \\
3 & 2 & & \cdots & 2 & 3 & 1 & \infty \\
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$\rightsquigarrow$ The associated Artin-Tits group is a locally Garside group of FC type.

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Definition

1. We fix a finite dimensional vector space V over a field $\mathbb{K}$. The Quantum Yang-Baxter Equation on V is the equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

of linear transformations on $V \otimes V \otimes V$ where the indeterminate is a linear transformation $R : V \otimes V \to V \otimes V$, and R^{ij} means R acting on the ith and jth components.

2. A set-theoretical solution of this equation is a pair (X, S) such that X is a basis for V, and $S : X \times X \to X \times X$ is a bijective map that induces a solution R of the QYBE.

$\rightsquigarrow$ the two main issues in the theory are (a) the construction of explicite solutions of the QYBE and (b) the classification of the solutions.

Garside groups and QYBE

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Definition

Let (X,S) be a set theoretical solution of the Yang-Baxter equation. We set $S(x,y) = (g_x(y), f_y(x))$.

(i) (X,S) is nondegenerate and symmetric if
  (a) the maps f_x and g_x are bijections
  (b) $S \circ S = Id_X$.
  (c) $S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}$.

(ii) A subset $Y \subseteq X$ is an invariant subset if $S(Y \times Y) = Y \times Y$.

(iii) (X,S) is said to be decomposable if X is a union of two nonempty disjoint invariant subsets.

Fact : (a) If Y is an invariant subset, then $(Y, S|_{Y\times Y})$ is a set theoretical solution of the Yang-Baxter equation.
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Garside groups and QYBE

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Definition

Assume (X,S) is non-degenerate and symmetric. The structure group of (X,S) is defined to be the group $G(X,S)$ with the following group presentation :

$$\langle X \mid xy = zt \text{ when } S(x,y) = (z,t)\rangle$$

Theorem (Chouraqui)

Assume (X,S) is non-degenerate and symmetric.

The structure group $G(X,S)$ is a Garside group.

The associated Garside monoid has the same presentation, considered as a monoid presentation.

Garside groups and QYBE

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Garside groups and QYBE
foldable solution

$\rightsquigarrow$ John Crisp has introduced the notion of a folding of a Coxeter graph (ie Coxeter matrix) and of an LCM-homomorphism. These notion can be used to prove that some subgroups of an Artin-Tits group are Artin-Tits groups.
$\rightsquigarrow$ (G) The notion of a subgroup obtained as the image of an LCM-homomorphism can be extended to the framework of Garside groups.
$\rightsquigarrow$ The notion of folding can be translated to the context of set theoretical solutions of the Yang-Baxter equation and should be seen as a generalization of the notion of decomposable solution :

Theorem (Chouraqui,G)

Let X be a finite set, and (X, S) be a non-degenerate, symmetric set-theoretical solution of the QYBE. The pair (X, S) is decomposable if and only if it has a strong folding (X', S') which a is trivial solution and such that $\#X' = 2$.

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Garside groups and QYBE

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Garside groups and QYBE

foldable solutions : example

Consider the group $G(X,S)$ where $X = \{x_1, x_2, x_3, x_4\}$ and the defining relations are

$$\begin{array}{lll} x_1^2 = x_2^2; & x_1x_2 = x_3x_4; & x_1x_3 = x_4x_2; \\ x_3^2 = x_4^2; & x_2x_4 = x_3x_1; & x_2x_1 = x_4x_3. \end{array}$$

$\rightsquigarrow$ The group $G(X,S)$ has no proper standard parabolic subgroup.

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$\rightsquigarrow$ Set $G(X',S') = \langle x,y \mid x^2 = y^2 \rangle$. Then,

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The theory of Algebraic monoids has been mainly developped by M. Pucha, L. Renner and L. Solomon. As remarked by the latter[1]

"Their work has been more or less ignored by those who might enjoy it and profit from it[...] This subjet has a marketing problem. My estimate, based on some minimal evidence, is that those who do algebraic groups are sympathetic but uninterested, those who do algebraic combinatorics do not know that the subject exists, and those who do semigroups are put off by prerequisites which seem formidable."

[1] Louis Solomon, An introduction to reductive monoids. Semigroups, Formal languages and groups, 295-352, J. Fountain (eds) Kluwer Acad. Publ., Dordrecht, 1995

Renner monoids

Algebraic monoids

Definition

Let $\mathbb{K}$ be an algebraic closed field. An *algebraic monoid* M is a submonoid (for product) of some $M_n(\mathbb{K})$ that is closed for the topology of Zariski. A *reductive monoid* is a irreducible algebraic monoid whose unit group is a reductive group.

Definition

Let M be a reductive monoid. Let T be a maximal torus of G. Then, the *Renner monoid* of M is the monoid

$$R = \overline{N_G(T)}/T$$

Example

$M = M_n(\mathbb{K})$; $G = GL_n(\mathbb{K})$; $T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}$ The monoid R is the *Rook monoid* RS_n that is the monoid of partial permutations on n letters.

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Lemma

R is an finite factorizable inverse monoid. Its unit group is the Weyl group W of G. In particular we have

$$R = E(R) \cdot W$$

where $E(R) = \{e \in R \mid e^2 = e\}$. Furthermore, $E(R) = E(\overline{T})$ is a commutative monoid.

Remarks : (1) If $R = RS_n$ then, $W = S_n$, the permutation group.
(2) Weyl groups are Coxeter groups. They are classified and have presentation of the following type : $\left\langle S \mid s^2 = 1;\ \underbrace{sts\dots}_{m_{s,t}} = \underbrace{tst\dots}_{m_{s,t}} \quad s \neq t \right\rangle$.

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$$S_n = \left\langle s_1, \cdots s_{n-1} \mid s_i^2 = 1\ ;\ s_is_j = s_js_i \quad |i-j| > 1\ ;\ s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \right\rangle$$

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One can associate to each Coxeter group W an Artin-Tits group A by removing the torsion relation from the presentation. $A = \langle S \mid \underbrace{sts\ldots}_{m_{s,t}} = \underbrace{tst\ldots}_{m_{s,t}} \quad s \neq t \rangle$

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If $W = S_n$, then A is the braid group B_n on n strings.

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One can associate to each Coxeter group W a generic Hecke algebra $\mathcal{H}(W)$ over $\mathbb{Z}[q, q^{-1}]$ so that

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Weyl groups

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Assume $\mathbb{K} = \overline{\mathbb{F}}_q$ and set $\varepsilon = \frac{1}{|\mathbb{B}_q|}\sum_{b\in\mathbb{B}_q} b$ in $\mathbb{C}[\mathbb{G}_q]$. The Iwahori-Hecke algebra $\mathcal{H}(\mathbb{G}_q,\mathbb{B}_q)$ is defined by

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$\mathcal{H}(\mathbb{G}_q,\mathbb{B}_q)$ is isomorphic to $\oplus_{w\in W}\mathbb{C}w$ as a $\mathbb{C}$ vector space. Moreover,

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The Iwahori-Hecke algebra $\mathcal{H}(\mathbb{G}_q,\mathbb{B}_q)$ is isomorphic to $\mathcal{H}_q(W)\otimes_{\mathbb{Z}}\mathbb{C}$, where $\mathcal{H}_q(W)$ is the specialisation at q of the generic Hecke algebra $\mathcal{H}(W)$ of the we group W.

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Assume $\mathbb{K} = \overline{\mathbb{F}}_q$ and set $\varepsilon = \frac{1}{|\mathbb{B}_q|}\sum_{b\in\mathbb{B}_q} b$ in $\mathbb{C}[\mathbb{M}_q]$. The monoid $\mathbb{M}_q$ is called a finite reductive monoid. The Iwahori-Hecke algebra $\mathcal{H}(\mathbb{M}_q, \mathbb{B}_q)$ is $\varepsilon\mathbb{C}[\mathbb{M}_q]\varepsilon$. It is isomorphic to $\oplus_{r\in R}\mathbb{C}r$ as a $\mathbb{C}$ vector space.

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- (L. Solomon) Does there exist a generic Hecke algebra $\mathcal{H}(R)$ so that

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W is a Coxeter group, and for $w \in W$ we have $\ell_S(w) = \inf\{k \mid w = s_1 \cdots s_k\}$

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$$A = \langle \underline{w}, w \in W \mid \underline{w}\,\underline{w'} = \underline{ww'} \text{ when } \ell_S(w) + \ell_S(w') = \ell_S(ww') \rangle$$

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(a) Is there *good* presentations for Renner monoids ?

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Weyl groups and length function

W is a Coxeter group, and for $w \in W$ we have $\ell_S(w) = \inf\{k \mid w = s_1 \cdots s_k\}$

Lemma

$$A = \langle \underline{w}, w \in W \mid \underline{w}\,\underline{w'} = \underline{ww'} \textit{ when } \ell_S(w) + \ell_S(w') = \ell_S(ww') \rangle$$

$$\mathcal{H}(W) = \left\langle T_w, w \in W \;\middle|\; \begin{array}{ll} T_w T_{w'} = T_{ww'} & \ell_S(w) + \ell_S(w') = \ell_S(ww') \\ T_s^2 = (q-1)T_s + qT_1 & s \in S \end{array} \right\rangle$$

$$\mathbb{B}s\mathbb{B}w\mathbb{B} = \begin{cases} \mathbb{B}sw\mathbb{B} & \textit{if } \ell(sw) = \ell(w) + 1; \\ \mathbb{B}sw\mathbb{B} \cup \mathbb{B}w\mathbb{B} & \textit{if } \ell(sw) = \ell(w) - 1. \end{cases}$$

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Presentation of Renner monoids
Generating set

$$R = E(R) \cdot W.$$

$\Rightarrow$ If Λ is a transversal of $E(R)$ under the action of W, then $S \cup \Lambda$ is a generating set of R.

Theorem

Let $\mathbb{B}$ be a Borel subgroup of the algebraic group G, and let T be a maximal torus in $\mathbb{B}$. Then,

$$\Lambda(\mathbb{B}) = \{e \in E(R) \mid \forall b \in \mathbb{B},\ ebe = eb\}$$

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Presentation of Renner monoids

Generating set of the Rook monoid

Example

$$M = M_n(\mathbb{K})\,;\, G = GL_n(\mathbb{K})\,;\, \mathbb{B} = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix};\, T = \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix}.$$

One has $R = RS_n$ and

$$\Lambda(\mathbb{B}) = \left\{ e_0 = (0), e_1 = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix}, \ldots, e_{n-1} = \begin{pmatrix} I_{n-1} & 0 \\ 0 & 0 \end{pmatrix}, e_n = Id_n \right\}$$

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Lemma (G)

Let $\Omega = \{e_0, \ldots, e_{n-1}, s_1, \ldots, s_{n-1}\}$. Then, the Rook monoid has a monoid presentation whose generating set is Ω and whose defining relations are :

$$\text{braid relations :} \quad \begin{cases} s_i^2 &= 1 \\ s_i s_j &= s_j s_i \qquad |i-j| \geq 2 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \end{cases}$$

$$\text{relations in } E(RS_n) : \quad e_i e_j = e_j e_i = e_{\min(i,j)}$$

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Presentation of Renner monoids

Defining relations of the Rook monoid

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Let R be a Renner monoid with genereting set $S \cup \Lambda_\circ$ Then the following relations provided a monoid presentation of R.

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where $\underline{w}$ is an arbitrary representative word of w.

This result holds for

⇝ Renner monoids of reductive monoids and finite reductive monoids.

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Length on the rook monoids

Definition and first properties

Definition (G)

(i) We set $\ell(s_i) = 1$ and $\ell(e_j) = 0$. For a word $\rho = x_1 \cdots x_k$ with $x_1, \dots, x_k$ in $S \cup \Lambda_\circ$, we set

$$\ell(\rho) = \sum_{i=1}^{k} \ell(x_i)$$

(ii) For r in R, we set : $\ell(r) = \min\{\ell(\rho), \rho \in (S \cup \Lambda_\circ)^* \mid \overline{\rho} = r\}$.

Lemma

(i) $\ell|_W = \ell_S$ *; if* $\ell(w) = 0$ *then* $w \in \Lambda$*.*

(ii) If $s \in S$ *then* $|\ell(sw) - \ell(w)| \leq 1$*.*

(iii) $\ell(ww') \leq \ell(w) + \ell(w')$*.*

(iv) If $r = w_1 e w_2$ *is the normal decomposition of* r *(*$e \in \Lambda$*,* $w_1, w_2 \in W$*) then*

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Let r be in R and $r = w_1 e w_2$ its normal decomposition of r. Then,

$$\ell(r) = \dim(\mathbb{B} w_1 e \mathbb{B}) - \dim(\mathbb{B} e w_2 \mathbb{B}).$$

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Iwahori-Hecke algebra

Theorem (Solomon/Pennel-Putcha-Renner, G)

Let M be a finite reductive monoid over $\overline{\mathbb{F}}_q$. The Iwahori-Hecke algebra $\mathcal{H}(M,B)$ has the following $\mathbb{C}$-algebra presentation :

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Theorem (G)

To each generalized Renner monoid R, one can associate a generic Hecke algebra $\mathcal{H}(R)$ that is a ring over the free $\mathbb{Z}[q]$-module with base R.
$\rightsquigarrow$ We obtain a positive answer to Solomon question :
$\mathcal{H}(\mathbb{M}_q,\mathbb{B}_q) = \mathcal{H}_q(R) \otimes_{\mathbb{Z}} \mathbb{C}$.

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