Algebraic generalization of braid groups

Eddy Godelle

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Caen 7 avril 2011

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- 2. The parabolics of Garside structure.
- 3. Hecke Algebra of Renner monoids.
- 4. Free group automorphisms and braid representation.

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Example

Let $M = (m_{s,t})_{s,t \in S}$ be a Coxeter matrix.

$$A = \left\langle S \mid \underbrace{sts...}_{m_{s,t}} = \underbrace{tst...}_{m_{s,t}} \quad s \neq t \right\rangle$$

The group *A* is called the Artin-Tits group associated with *M*.

→→ The submonoid A^+ generated by *S* as the same presentation. →→ The associated Coxeter group is $W = A/_{s^2=1}$.

If $S = \{s_1, \dots, s_n\}$ with $m_{s_i,s_j} = 3$ for |i - j| = 1 and $m_{s_i,s_j} = 2$ for |i - j| > 1, the associated Artin-Tits group is the braid group B_{n+1} on n+1 strands. $W = \mathfrak{S}_{n+1}$.

$$\Delta_n = (\sigma_1 \cdots \sigma_{n-1} \sigma_n) \cdots (\sigma_1 \sigma_2) \sigma_1$$

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Definition

(i) A monoid is said to be a locally Garside monoid if

- (a) it is cancellative and Noetherian ;
- (b) any two elements have a common multiple for left-divisibility if and only if they have a least common multiple for left-divisibility;
- (c) any two elements have a common multiple for right-divisibility if and only if they have a least common multiple for right-divisibility.

(ii) A Garside element of a locally Garside monoid is a balanced element whose set of factors generates the whole monoid. When such an element exists, we say that the monoid is a Garside monoid.

(iii) A (locally) Garside group G(M) is the enveloping group of a (locally) Garside monoid M.

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parabolic subgroups of Artin-Tits groups

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\rightsquigarrow the associated Garside monoid is A^+ .

Theorem

Let $T \subseteq S$ and $N = (m_{s,t})_{s,t \in T}$

$$A_T := \langle T \rangle_A \simeq \left\langle T \mid \underbrace{sts...}_{m_{s,t}} = \underbrace{tst...}_{m_{s,t}} \quad s \neq t \right\rangle.$$

The subgroup A_T is called a standard parabolic subgroup of A_{T} , $A_$

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\rightsquigarrow A_T is an Artin-Tits group;

- \rightsquigarrow The Garside structures of A and A_T are compatibles ;
- $\rightsquigarrow A^+ \cap A_T = A_T^+;$
- $\rightsquigarrow A_T \cap A_U = A_{T \cap U};$
- The family of parabolic subgroups allows to solve the word problem in A (for some cases).
- The family of parabolic subgroups allows to build a finite dimensional CAT(0) complex on which A acts (for some cases).
- → The family of parabolic subgroups allows to prove the $K(\pi, 1)$ conjecture for Coxeter groups (for some cases).

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Question : Is there a natural notion of parabolic subgroup in the framework of Garside groups ? Can we define a standard parabolic subgroup as any subgroup generated by a subset of atoms ?

Question : Is there a natural notion of Coxeter-like associated group in the framework of Garside groups ? Remark : General Artin-Tits groups are badly understood (not even a solution to the word problem). So, one may wonder :

Why is it interesting to consider locally Garside groups which are more general ?

A better abstract understanding can help to understand Artin-Tits groups.
 Natural objects of study can be equiped with a Garside structure.
 Garside groups are well-understood, as spherical-type Artin-Tits groups.
 Some Artin-Tits groups are (locally) Garside groups and can be understood in this way (word problem...).

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Garside groups Two questions and a remark

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→ First step : What about Garside groups ?

Example

Consider $M = \langle a_1, a_2, a_3, a_4 \mid a_1a_2 = a_2a_3 = a_3a_4 = a_4a_1 \rangle$. The subgroup $G_{1,3}$ is a free group with base $\{a_1, a_3\}$, and $G_{1,3} \cap M$ is the free monoid with base $\{a_1, a_3\}$.

$$G_{1,3}\cap G_{2,4}=\langle a_3^{-1}a_1\rangle.$$

Lemma

If *M* is an Artin-Tits monoid of spherical type, then its classical parabolic submonoids are in one-to-one correspondance with the balanced elements that belongs to $Div(\Delta)$.

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$$\mathbf{G}_{1,3}\cap\mathbf{G}_{2,4}=\langle \mathbf{a}_3^{-1}\mathbf{a}_1\rangle.$$

Lemma

If *M* is an Artin-Tits monoid of spherical type, then its classical parabolic submonoids are in one-to-one correspondance with the balanced elements that belongs to $Div(\Delta)$.

- 21

Definition (G)

Let *M* be a Garside monoid and Δ be a Garside element. If $\delta \in Div(\Delta)$ balanced, then M_{δ} is a parabolic submonoid with δ as a Garside element if and only if

 $\operatorname{Div}(\delta) = \operatorname{Div}(\Delta) \cap M_{\delta}.$

Definition (G-Paris)

Let *M* be a monoid and *N* be a submonoid. We say that *N* is special when it is closed by factors, that is $abc \in N \implies a, b, c \in N$.

Lemma

Let *M* be a locally Garside monoid and *N* be a special submonoid. Then *N* is a locally Garside monoid, and a lower sub-semilattice.

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Definition (G-Paris)

Let *M* be a Garside monoid and G(M) be its associated Garside group.

- (i) A submonoid of *M* is said to be parabolic if it is special, and closed by left lcm and by right lcm. A parabolic submonoid is of spherical type if it has a Garside element.
- (ii) A subgroup of G(M) is standard parabolic if it is generated by the image t(N) of a parabolic submonoid N of M.

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Let M be a Garside monoid.

Every parabolic submonoid N of M is of spherical type. Moreover, there exists a Garside element Δ_N of N such that

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Conjecture (G-Paris)

For every locally Garside monoid, Properties (P1), (P2), (P3) and (P4) hold.

- (P1) The canonical morphism $\iota: M \to G(M)$ is into.
- (P2) The group G(M) is torsion free.
- (P3) If N is a parabolic submonoid, then the associated standard parabolic subgroup is isomorphic to G(N). Moreover, $G(N) \cap M = N$ in G(M).

(P4) If N and N' are parabolic submonoids then $N \cap N'$ is parabolic and $G(N \cap N') = G(N) \cap G(N')$.

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Definition (G-Paris)

The family of locally Garside groups of FC type is the smallest family of locally Garside groups that contains Garside groups and that is closed by amalgamation above a parabolic subgroup.

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Definition

We fix a finite dimensional vector space V over a field K. The Quantum Yang-Baxter Equation on V is the equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$$

of linear transformations on $V \otimes V \otimes V$ where the indeterminate is a linear transformation $R: V \otimes V \rightarrow V \otimes V$, and R^{ij} means R acting on the *i*th and *j*th components.

② A set-theoretical solution of this equation is a pair (X, S) such that X is a basis for V, and S : X × X → X × X is a bijective map that induces a solution R of the QYBE.

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nondegenerated ans symmetric solutions

Definition

Let (X, S) be a set theoretical solution of the Yang-Baxter equation. We set $S(x, y) = (g_x(y), f_y(x)).$

(i) (X, S) is nondegenerate and symmetric if

(a) the maps f_x and g_x are bijections

(b)
$$S \circ S = Id_X$$
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$$S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}$$
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(ii) A subset $Y \subseteq X$ is an invariant subset if $S(Y \times Y) = Y \times Y$.

(iii) (X, S) is said to be decomposable if X is a union of two nonempty disjoint invariant subsets.

Fact : (a) If Y is an invariant subset, then $(Y, S|_{Y \times Y})$ is a set theoretical solution of the Yang-Baxter equation.

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Garside groups and QYBE structure groups

Definition

Assume (X, S) is non-degenerate and symmetric. The structure group of (X, S) is defined to be the group G(X, S) with the following group presentation :

 $\langle X \mid xy = zt \text{ when } S(x, y) = (z, t) \rangle$

Theorem (Chouraqui)

Assume (X, S) is non-degenerate and symmetric.

The structure group G(X, S) is a Garside group.

The associated Garside monoid has the same presentation, considered as a monoid presentation.

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invariant subset and parabolic subgroups

Theorem (Chouraqui,G)

For $Y\subseteq X,$ denote by G_Y the subgroup of G(X,S) generated by Y. The map

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invariant subsets of (X, S)

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→ John Crisp has introduced the notion of a folding of a Coxeter graph (ie Coxeter matrix) and of an LCM-homomorphism. These notion can be used to

prove that some subgroups of an Artin-Tits group are Artin-Tits groups. →→ (G) The notion of a subgroup obtained as the image of an LCM-homomorphism can be extended to the framework of Garside groups. →→ The notion of folding can be translated to the context of set theoretical solutions of the Yang-Baxter equation and should be seen as a generalization of the notion of decomposable solution :

Theorem (Chouraqui,G)

Let X be a finite set, and (X, S) be a non-degenerate, symmetric set-theoretical solution of the QYBE. The pair (X, S) is decomposable if and only if it has a strong folding (X', S') which a is trivial solution and such that #X' = 2.

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foldable solutions : example

Consider the group G(X, S) where $X = \{x_1, x_2, x_3, x_4\}$ and the defining relations are

$$\begin{array}{ll} x_1^2 = x_2^2; & x_1 x_2 = x_3 x_4; & x_1 x_3 = x_4 x_2; \\ x_3^2 = x_4^2; & x_2 x_4 = x_3 x_1; & x_2 x_1 = x_4 x_3. \end{array}$$

→ The group G(X, S) has no proper standard parabolic subgroup. → The solution (X, S) is not decomposable. → Set $G(X', S') = \langle x, y | x^2 = y^2 \rangle$. Then,

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Here $X' = \{x = x_1^2, y = x_3^2\}$, and the sets $\{x_1, x_2\}$ and $\{x_3, x_4\}$ generate Garside subgroups that are not parabolic.

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The theory of Algebraic monoids has been mainly developped by M. Pucha, L. Renner and L. Solomon. As remarked by the latter¹

"Their work has been more or less ignored by those who might enjoy it and profit from it[...] This subjet has a marketing problem. My estimate, based on some minimal evidence, is that those who do algebraic groups are sympathetic but uninterested, those who do algebraic combinatorics do not know that the subject exists, and those who do semigroups are put off by prerequisites which seem formidable."

¹Louis Solomon, An introduction to reductive monoids. Semigroups, Formal languages and groups, 295-352, J. Fountain (eds) Kluwer Acad. Publ., Dordrecht, 1995

Algebraic monoids

Definition

Let \mathbb{K} be an algebraic closed field. An *algebraic monoid* M is a submonoid (for product) of some $M_n(\mathbb{K})$ that is closed for the topology of Zariski. A *reductive monoid* is a irreducible algebraic monoid whose unit group is a reductive group.

Definition

Let *M* be a reductive monoid. Let *T* be a maximal torus of *G*. Then, the *Renner monoid* of *M* is the monoid

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Example $M = M_n(\mathbb{K})$; $G = GL_n(\mathbb{K})$; $T = \begin{pmatrix} * \\ & * \\ & * \end{pmatrix}$ The monoid *R* is the *Rook monoid RS_n* that is the monoid of partial permutations on *n* letters.

Algebraic monoids

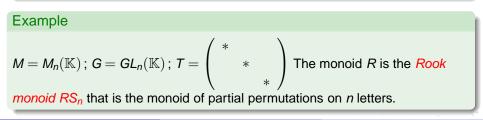
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Renner monoids and Weyl groups

Lemma

R is an finite factorizable inverse monoid. Its unit group is the Weyl group W of G. In particular we have

$$R = E(R) \cdot W$$

where $E(R) = \{e \in R \mid e^2 = e\}$. Furthermore, $E(R) = E(\overline{T})$ is a commutative monoid.

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$$S_n = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1; s_i s_j = s_j s_i \mid i-j| > 1; s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \rangle$$

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Remarks : (1) If $R = RS_n$ then, $W = S_n$, the permutation group.

(2) Weyl groups are *Coxeter groups*. They are classified and have presentation of the following type : $\left\langle S \mid s^2 = 1; \underbrace{sts...}_{m_{s,t}} = \underbrace{tst...}_{m_{s,t}} s \neq t \right\rangle$.

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Weyl groups Artin-Tits groups and generic Hecke algebra

Definition

One can associate to each Coxeter group *W* an Artin-Tits group *A* by removing the torsion relation from the presentation. $A = \langle S \mid \underline{sts...} = \underline{tst...} \quad s \neq t \rangle$

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Assume $\mathbb{K} = \overline{\mathbb{F}}_q$ and set $\varepsilon = \frac{1}{|\mathbb{B}_q|} \sum_{b \in \mathbb{B}_q} b$ in $\mathbb{C}[\mathbb{G}_q]$. The Iwahori-Hecke algebra $\mathcal{H}(\mathbb{G}_q, \mathbb{B}_q)$ is defined by

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The Iwahori-Hecke algebra $\mathcal{H}(\mathbb{G}_q, \mathbb{B}_q)$ is isomorphic to $\mathcal{H}_q(W) \otimes_{\mathbb{Z}} \mathbb{C}$, where $\mathcal{H}_q(W)$ is the specialisation at q of the generic Hecke algebra $\mathcal{H}(W)$ of the we group W.

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Generic Hecke algebra and Iwahori-Hecke algebra

Assume $\mathbb{K} = \overline{\mathbb{F}}_q$ and set $\varepsilon = \frac{1}{|\mathbb{B}_q|} \sum_{b \in \mathbb{B}_q} b$ in $\mathbb{C}[\mathbb{M}_q]$. The monoid \mathbb{M}_q is called a finite reductive monoid. The Iwahori-Hecke algebra $\mathcal{H}(\mathbb{M}_q, \mathbb{B}_q)$ is $\varepsilon \mathbb{C}[\mathbb{M}_q]\varepsilon$. It is isomorphic to $\bigoplus_{r \in R} \mathbb{C}r$ as a \mathbb{C} vector space. Questions :

• (L. Solomon) Does there exist a generic Hecke algebra $\mathcal{H}(R)$ so that

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W is a Coxeter group, and for $w \in W$ we have $\ell_{S}(w) = \inf\{k \mid w = s_{1} \cdots s_{k}\}$

Lemma

$$A = \left\langle \underline{w}, w \in W \mid \underline{w} \, \underline{w'} = \underline{ww'} \text{ when } \ell_{\mathcal{S}}(w) + \ell_{\mathcal{S}}(w') = \ell_{\mathcal{S}}(ww') \right\rangle$$

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(a) Is there good presentations for Renner monoids?

(b) Does there exists a good length function on Renner monoids that do the job?

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W is a Coxeter group, and for $w \in W$ we have $\ell_{S}(w) = \inf\{k \mid w = s_{1} \cdots s_{k}\}$

Lemma

$$A = \left\langle \underline{w}, w \in W \mid \underline{w \; w'} = \underline{ww'} \; when \; \ell_{\mathcal{S}}(w) + \ell_{\mathcal{S}}(w') = \ell_{\mathcal{S}}(ww') \right\rangle$$

$$\mathcal{H}(W) = \left\langle T_{w}, w \in W \mid \begin{array}{cc} T_{w} T_{w'} = T_{ww'} & \ell_{S}(w) + \ell_{S}(w') = \ell_{S}(ww') \\ T_{s}^{2} = (q-1)T_{s} + qT_{1} & s \in S \end{array} \right\rangle$$
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Presentation of Renner monoids Generating set

 $R = E(R) \cdot W.$

⇒ If Λ is a transversal of E(R) under the action of W, then $S \cup \Lambda$ is a generating set of R.

Theorem

Let $\mathbb B$ be a Borel subgroup of the algebraic group G, and let T be a maximal torus in $\mathbb B$. Then,

$$\Lambda(\mathbb{B}) = \{ e \in E(R) \mid \forall b \in \mathbb{B}, ebe = eb \}$$

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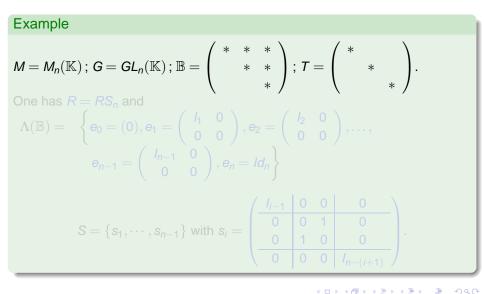
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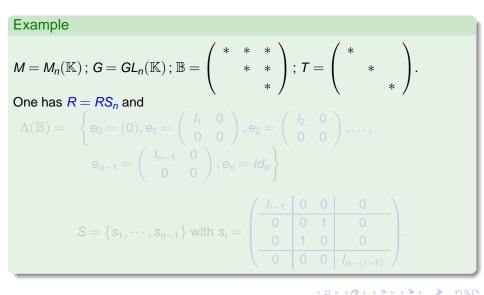
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Generating set of the Rook monoid

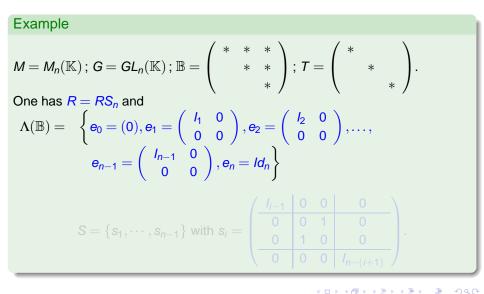


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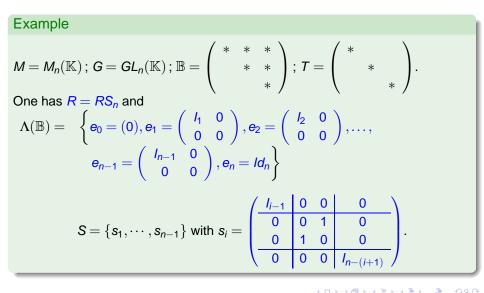


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Generating set of the Rook monoid



Generating set of the Rook monoid



Defining relations of the Rook monoid

Lemma (G)

Let $\Omega = \{e_0, \dots, e_{n-1}, s_1, \dots, s_{n-1}\}$. Then, the Rook monoid has a monoid presentation whose generating set is Ω and whose defining relations are :

braid relations :
$$\begin{cases} s_i^2 = 1\\ s_i s_j = s_j s_i \\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \end{cases} |i-j| \ge 2$$

relations in $E(RS_n)$: $e_i e_j = e_j e_i = e_{\min(i,j)}$

$$S_n \text{ and } E(RS_n): \begin{cases} e_j s_i = s_i e_j & i < j \\ e_j s_i = s_i e_j = e_j & j < i \\ e_i s_i e_i = e_{i-1} \end{cases}$$

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where w is an arbitrary representative word of w.

This result holds for

→ Renner monoids of reductive monoids and finite reductive monoids.

- ~> Renner monoids of finite monoids of Lie type.
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Length on the rook monoids

Definition and first properties

Definition (G)

(i) We set $\ell(s_i) = 1$ and $\ell(e_j) = 0$. For a word $\rho = x_1 \cdots x_k$ with x_1, \dots, x_k in $S \cup \Lambda_\circ$, we set

$$\ell(\rho) = \sum_{i=1}^{k} \ell(\mathbf{x}_i)$$

(ii) For r in R, we set : $\ell(r) = \min{\{\ell(\rho), \rho \in (S \cup \Lambda_{\circ})^* \mid \overline{\rho} = r\}}.$

Lemma

(i) $\ell|_W = \ell_S$; if $\ell(w) = 0$ then $w \in \Lambda$. (ii) If $s \in S$ then $|\ell(sw) - \ell(w)| \le 1$. (iii) $\ell(ww') \le \ell(w) + \ell(w')$. (iv) If $r = w_1 e w_2$ is the normal decomposition of $r \ (e \in \Lambda, w_1, w_2 \in W)$ then

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Length on the rook monoid Main properties

Theorem (G)

Let r be in R and $r = w_1 e w_2$ its normal decomposition of r. Then,

 $\ell(r) = \dim(\mathbb{B}w_1 e \mathbb{B}) - \dim(\mathbb{B}ew_2 \mathbb{B}).$

 $\mathbb{B}s\mathbb{B}r\mathbb{B} = \begin{cases} \mathbb{B}r\mathbb{B} & \text{if } \ell(sr) = \ell(r);\\ \mathbb{B}sr\mathbb{B} & \text{if } \ell(sr) = \ell(r) + 1;\\ \mathbb{B}sr\mathbb{B} \cup \mathbb{B}r\mathbb{B} & \text{if } \ell(sr) = \ell(r) - 1. \end{cases}$

Iwahori-Hecke algebra

Theorem (Solomon/Pennel-Putcha-Renner, G)

Let *M* be a finite reductive monoid over $\overline{\mathbb{F}}_q$. The Iwahori-Hecke algebra $\mathcal{H}(M, B)$ has the following \mathbb{C} -algebra presentation :

 $\begin{array}{ll} (\text{HEC1}) & T_s^2 = (q-1)T_s + qT_1, & s \in S \ ; \\ (\text{HEC2}) & \underbrace{T_sT_tT_s\cdots}_{m_{s,t} \ termes} = \underbrace{T_tT_sT_t\cdots}_{m_{s,t} \ termes}, & s,t \in S \ ; \\ (\text{HEC3}) & T_sT_e = T_eT_s, & e \in \Lambda_\circ, s \in \lambda^*(e) \ ; \\ (\text{HEC4}) & T_sT_e = T_eT_s = qT_e, & e \in \Lambda_\circ, s \in \lambda_*(e) \ ; \\ (\text{HEC5}) & T_eT_wT_f = q^{\ell(w)}T_{e\wedge wf}, & e,f \in \Lambda_\circ, w \in Red(e,f). \end{array}$

Theorem (G)

To each generalized Renner monoid R, one can associate a generic Hecke algebra $\mathcal{H}(R)$ that is a ring over the free $\mathbb{Z}[q]$ -module with base R. \rightsquigarrow We obtain a positive answer to Solomon question : $\mathcal{H}(\mathbb{M}_q, \mathbb{B}_q) = \mathcal{H}_q(R) \otimes_{\mathbb{Z}} \mathbb{C}.$

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