Garside Categories

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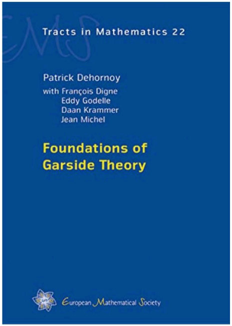
Garside categories

- Lecture 1 : Do Garside categories exist and why should we care about it ?
- Lecture 2 : Working with Garside families in categories and groupoid.
- Lecture 3 : Existence of Garside families - Quasi-Garside category.

References

1. The lecture is based on the reference Book : Foundations of Garside Theory, P. Dehornoy, F. Digne, E. Godelle D. Krammer et J. Michel, *Europ. Math. Soc. Tracts in Mathematics* 22.



2. The book does not contain any reference to a notion of a Garside category.
3. Only small categories are considered, and are meanly seen as a way to encode a partial product between arrows.

Lecture 1 :
Do Garside categories exist and why should we care about it ?

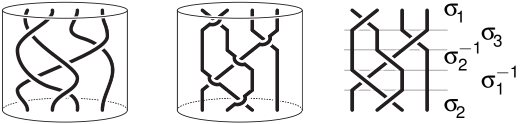
1. Introduction and motivations to Garside framework
2. Why consider category/groupoid and not just monoid/group ?
3. Greedy and normal decomposition : Garside family

Introduction and motivations

Braid groups

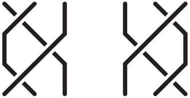
The braid group on n strands B_n admits the following presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ccc} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} & |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} & |i-j| = 1 \end{array} \right\rangle \tag{1}$$



$\sigma_1\sigma_3 \equiv \sigma_3\sigma_1$

$\sigma_1\sigma_1^{-1} \equiv \varepsilon \equiv \sigma_1^{-1}\sigma_1$



$\sigma_1\sigma_2\sigma_1 \equiv \sigma_2\sigma_1\sigma_2$

Introduction and motivations

Braid groups

The braid group on n strands B_n admits the following presentation

$$\left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{cc} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} \quad |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} \quad |i-j| = 1 \end{array} \right\rangle \tag{2}$$

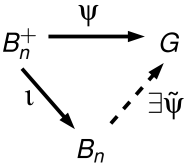
1. In his PHD Thesis (1965), Garside solved the word problem and the conjugacy problem for braid groups .
2. He used the positive braid monoid B_n^+ that possesses the same presentation as B_n, but considered as a presentation of monoid.
3. Why is it easier consider B_n^+ than B_n ?
    1. The relations of the presentation are homogeneous and pre-cancellative.
    2. Left and right divisibilities are partial order, there is atoms, every element has a finite set of divisors, one can do induction...
    3. For instance : solving the words problem is obvious : starding from any (positive) word, one can write all the (positive) words that represent the same element in B_n^+.

Introduction and motivations

Braid groups

Question : What is the connection between B_n and B_n^+ ?

1. There is an obvious morphism (of monoids) $\iota : B_n^+ \to B_n, \sigma_i \mapsto \sigma_i$.
2. Indeed B_n is the enveloping group of B_n^+.



Problem : In general the canonical morphim $M \to Env(M)$ is not into : the enveloping group of the monoid $\langle a \mid a^2 = a \rangle$ is the trivial group.

1. If $M \to Env(M)$ is into then M is cancellative ($abc = ab'c \Rightarrow b = b'$).
2. cancellativity is not a sufficient condition for embedding.

Example : Consider

$$M = \left\langle a, b, c, d, e \;\middle|\; \begin{array}{l} ab = cd \\ aeb = ced \end{array} \right\rangle$$

1. The monoid M is cancellative (prove it ! hint :defining relations does not overlap) ?
2. $ae^2b \neq ce^2d$ in M (no defining relation can be applied to the words)
3. in $Env(M)$ one has

$$ae^2b = aebb^{-1}a^{-1}aeb = cedd^{-1}c^{-1}ced = ce^2d.$$

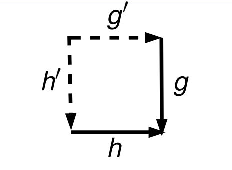
Theorem (left-Öre condition)

Assume M is a monoid so that

(a) *M is cancellative ;*

(b) *For any g, h in M there exists g', h' in M so that*

$$g'g = h'h$$



. Then M embeds in $Env(M)$ and for any element g of $Env(M)$ there exists g_1, g_2 in M so that $g = g_1^{-1}g_2$.

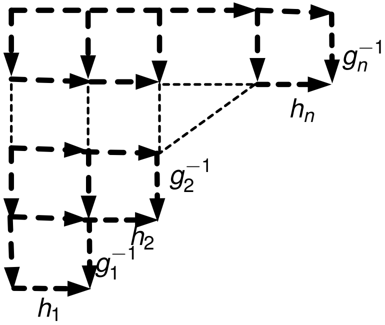
Remark

1. *Similarly, right-Öre conditions holds.*
2. *The formalism with arrows is in the spirit of categories. Indeed the result, and its proof extend verbatim to categories (replacing enveloping group with enveloping groupoid).*

Introduction and motivations

Braid groups - Öre criterium

Equality $g'g = h'h$ in the second condition becomes $h'^{-1}g' = hg^{-1}$ in $Env(M)$: this allows in any decomposition of an element to push all the inverse elements on the left, and explain why any element can be written as a fraction.

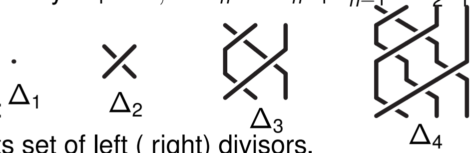


$$g = h_1 g_1^{-1} h_2 g_2^{-1} \cdots h_n g_n^{-1} = g'^{-1} g''$$

Introduction and motivations

Braid groups - Garside results

In order to solve the word problem and the conjugacy problem, in B_n Garside consider monoid of B_n^+ and the particular element (the so-called Garside element) Δ_n defined by $\Delta_1 = 1, \quad \Delta_n = \Delta_{n-1}\,\sigma_{n-1}\cdots\sigma_2\sigma_1$ for $n \geqslant 2$ which in B_n correspond to :



The main tool is the properties of its set of left (right) divisors.

1. Sets of left divisors and right divisors of Δ coincide (denoted by $\mathrm{Div}(\Delta_n)$).
2. $\mathrm{Div}(\Delta_n)$ generates B_n^+.
3. Any non-empty subset of $\mathrm{Div}(\Delta_n)$ has a unique left (right) lcm in $\mathrm{Div}(\Delta_n)$.

Introduction and motivations

Braid groups - Garside results

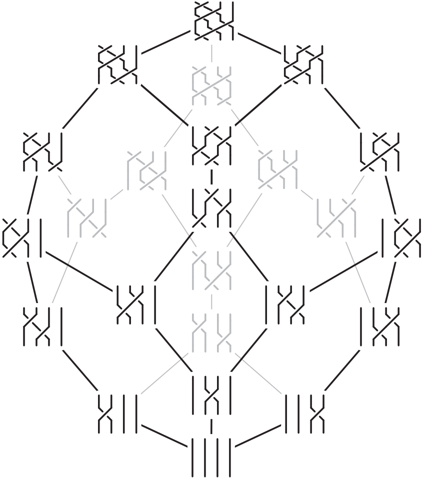


FIGURE – The lattice $(\mathrm{Div}(\Delta_4), \preceq)$ with the 24 left-divisors Δ_4 in the monoid B_4^+

By induction on the length of elements, Garside gets that

1. B_n^+ is cancellative ;
2. any two elements of B_n^+ possess a left (right) lcm in B_n^+.

In particular B_n^+ satisfies the (left and right) Öre conditions and can be identified with the submonoid od B_n^+ of positive braids.
Moreover Δ_n^2 is central and any element g of B_n can be written as $g = g^+\Delta_n^k$ with g^+ in B_n^+ and k in $\mathbb{Z}$.

Solution to the word problem : any signed word can be transformed into a product of a positive word and a negative word. Then we can verify whether the two words represent the same element in the monoid B_n^+.

Solution to the conjugacy problem :

Since Δ_n^2 is central and any element g of B_n can be written as $g = g^+\Delta_n^k$ with g^+ in B_n^+ and k in $\mathbb{Z}$, then it enough to solve the conjugacy problem in B_n^+. Let g g' be in B_n^+. We can assume $g \neq 1$. Associate to g the set $Conj_\infty(g)$ built in the following way :

1. $Conj_0(g) = \{g\}$.
2. $Conj_{i+1}(g) = Conj_i(g) \cup \{g'' = hg'h^{-1} \in B_n^+ \mid g' \in Conj_i(g); h \in \mathrm{Div}(\Delta_n)\}$. (The later set is finite and we can decided if an element is positive)
3. the processus as to stabilized to $Conj_\infty(g)$.

Now, g and g' are conjugated iff g' belongs to $Conj(g)$.

The main interesting point is the proof of the result :

1. Assume a is a positive element that conjugate g into g', that is $ga = ag'$
2. Write $a = a_1 \cdots a_r$ with $a_1, \ldots, a_r$ in $\mathrm{Div}(\Delta_n)$; $a_r \neq 1$ and a_i of maximal length so that $a_i \cdots a_r$ can be written as $a_i b_i$ with b_i in B_n^+.

Then for all i the element $(a_1 \cdots a_i)^{-1} g (a_1 \cdots a_i)$ belongs to B_n^+.

So a_i is the left gcd of $a_i \cdots a_r$ and Δ (then a_i is unique), and the decomposition $a = a_1 \cdots a_r$ define a normal form on B_n^+. One can prove that $a_1 \cdots a_i$ is the left lcm of a and Δ_n^i.

Introduction and motivations

Braid groups - Garside results

This normal form is not clearly introduced in Garside work but is implicte and it is clearly used in Deligne proof for spherical type Artin-Tits groups (Deligne, Les immeubles des groupes de tresses généralisées, Inventiones Math 17 (1972))

(4.24) *Lemme. Soient x, y dans G^+. Si x et y sont conjugués, il existe une suite $x_0 = x, x_1, \dots, x_n = y$ d'éléments de G^+ et une suite d'éléments w_i de W telle que*

$$x_{i+1}\, r(w_i) = r(w_i)\, x_i.$$

Soit $a \in G^+$ tel que $x a = a y$. Posons $a = r(w_0) \dots r(w_{n-1})$, w_i étant l'élément de W de longueur maximum tel que $r(w_i) \dots r(w_{n-1})$ s'écrive sous la forme $r(w_i)\, b_i$ avec $b_i \in G^+$ (4.15). Soit $a_i = r(w_0) \dots r(w_i)$. Nous prouverons que $x_i = a_i^{-1} x a_i$ est dans G^+, de sorte que les x_i et w_i répondent au problème.

It first explicitely defined by Adian in 1984 and the independently by El-Rifai/Morton on 1994.

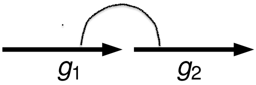
Introduction and motivations

Braid groups - Head

1. In the sequel, for g in B_n^+ we set $\alpha(g) = g \wedge \Delta_n$ (for the left divisibility.)
2. When g_1, g_2 lie in B_n^+ and we by $\alpha(g_1, g_2)$ we mean $\alpha(g_1 g_2)$. By $\omega(g_1, g_2)$ we de denote the element so that

$$g_1 g_2 = \alpha(g_1, g_2)\omega(g_1, g_2)$$

3. When g_1, g_2 lie in B_n^+ with $\alpha(g_1, g_2) = g_1$ (so g_1 belongs to $\mathrm{Div}(\Delta_n)$) we write :



4. Given an element, decomposed as a product of elements of $\mathrm{Div}(\Delta_n)$, what is the most efficient way to obtain its normal form ?

Introduction and motivations

Braid groups - Normal form

Example

Consider the element $g = \sigma_3\sigma_2{}^2\sigma_1\sigma_2{}^2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$ *in* B_4^+.

$g = \sigma_3\sigma_2 \cdot \sigma_2\sigma_1\sigma_2 \cdot \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$

$g = \sigma_3\sigma_2 \cdot \sigma_2\sigma_1\sigma_2 \cdot \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2 \cdot \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1 \cdot \sigma_2\sigma_3\sigma_2 =$
$\sigma_3\sigma_2 \cdot \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2 \cdot \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_1\sigma_2 \cdot \sigma_2\sigma_3\sigma_2 =$
$\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_1\sigma_2 \cdot \sigma_3\sigma_2\sigma_3 = \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_3 =$

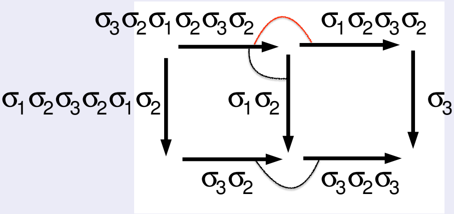


FIGURE – *The first domino rule :* $\alpha(g_1 g_2) = \alpha(g_1\alpha(g_2))$

Example

$g = \sigma_3\sigma_2 \cdot \sigma_1\sigma_2\sigma_1 \cdot \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1 = \sigma_3\sigma_2\sigma_1\sigma_2 \cdot \sigma_1 \cdot \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1 =$
$\sigma_3\sigma_2\sigma_1\sigma_2 \cdot \sigma_1 \cdot \sigma_2\sigma_1\sigma_3\sigma_2\sigma_1\sigma_3 = \sigma_3\sigma_2\sigma_1\sigma_2 \cdot \sigma_1\sigma_2\sigma_1\sigma_3\sigma_2\sigma_1 \cdot \sigma_3 =$
$\sigma_3\sigma_2\sigma_1\sigma_2 \cdot \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_3 = \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_3$

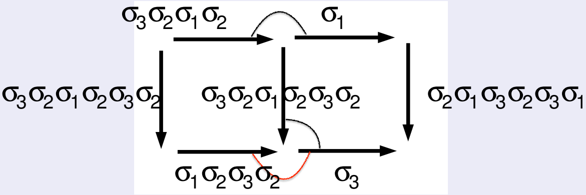


FIGURE – *The second domino rule :* $\alpha(g_1 g_2) = g_1 \Rightarrow \omega(g_1 g_2 g_3) = \omega(g_2 g_3)$

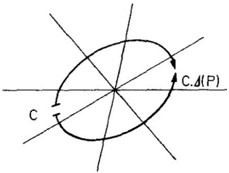
Introduction and motivations

Why consider category/groupoid ?

1. Garside theory extend to monoids that do not satisfied the Öre relations
2. A lot of definitions extends verbatim to the category context and some property are easier to visualized/understand/prove in this context.
3. Deligne did in when he extended the results of Garside to any Artin-Tits groups of spherical type.

(1.25) On peut regarder l'ensemble des galeries comme étant l'ensemble des flèches d'une catégorie $\mathrm{Gal}_0(V, \mathcal{M})$ ayant les chambres pour objets. De même, les classes d'équivalence de galeries sont les flèches d'une catégorie quotient $\mathrm{Gal}_+(V, \mathcal{M})$. Soient A, B et C trois chambres, E une galerie de A à B et F une galerie de B à C. Bien que les conventions générales dans les catégories soient autres, nous continuerons à noter EF le composé de E et F. La loi $*$ (resp. $G \to -G$) est une antiéquivalence (resp. une équivalence) de $\mathrm{Gal}_0(V, \mathcal{M})$ ou $\mathrm{Gal}_+(V, \mathcal{M})$ avec elle-même, induisant l'identité (resp. $C \to -C$) sur l'ensemble des objets.

Quand aucune confusion ne sera à craindre, nous écrirons simplement Gal_0 et Gal_+ pour $\mathrm{Gal}_0(V, \mathcal{M})$ et $\mathrm{Gal}_+(V, \mathcal{M})$ (ou pour une catégorie $\mathrm{Gal}_0(V_P, \mathcal{M}_P)$ ou $\mathrm{Gal}_+(V_P, \mathcal{M}_P)$). Par abus de langage, nous appelerons souvent *galeries* les flèches de Gal_+.



Introduction and motivations

Why consider category/groupoid ?

Example

Since Δ_4^2 is central in B_4. The normaliser $N(\sigma_2)$ of σ_2 in B_4 is generated (and is the envelopping group of) the monoid $N^+(\sigma_2)$ of its positive elements.
We have $\sigma_2(\sigma_1\sigma_2) = (\sigma_1\sigma_2)\sigma_1$ and $\sigma_1\sigma_3 = \sigma_3\sigma_1$. Then

1. $\sigma_1\sigma_2(\sigma_1\sigma_2\sigma_3)^{-1} \in N(\sigma_2)$ *but* $\sigma_1\sigma_2, \sigma_1\sigma_2\sigma_3 \notin N^+(\sigma_2)$.
2. $\sigma_1(\sigma_2)^2\sigma_1 \in N^+(\sigma_2)$ *and* $\sigma_1\sigma_2(\sigma_3)^2\sigma_2\sigma_1 \in N^+(\sigma_2)$ *; their left gcd is* $\sigma_1\sigma_2$ *and is not in* $N^+(\sigma_2)$.
3. $\sigma_1\sigma_2\sigma_3 \cdot \sigma_2\sigma_1 \in N^+(\sigma_2)$ *but its terms are not in* $N^+(\sigma_2)$

But $\sigma_2(\sigma_1\sigma_2) = (\sigma_1\sigma_2)\sigma_1$ and $\sigma_2(\sigma_1\sigma_2\sigma_3) = (\sigma_1\sigma_2\sigma_3)\sigma_1$.

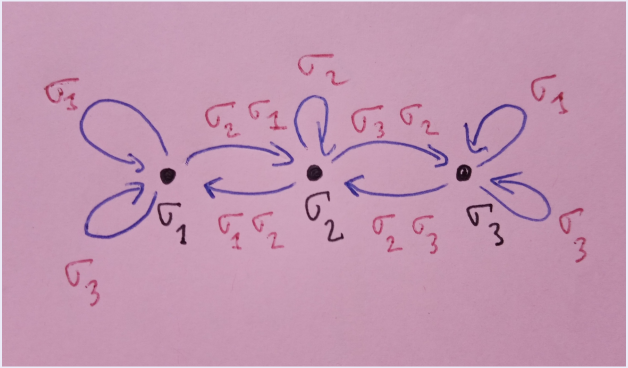
So in order to study the normaliser of a generator, we are lead to consider the groupoid (category) whom objects are the generators and whom morphisms are (positive) elements of B_4 that conjugate a generator on another. $\Rightarrow$ This is natural candidate to be a Garside groupoid (category).

Introduction and motivations

Why consider category/groupoid ?

Example

The category of quasi-centralisers of generators in B_4



$\sigma_2\sigma_1$ *lies in* $\mathscr{C}(\sigma_1,\sigma_2)$ *since* $\sigma_1(\sigma_2\sigma_1) = (\sigma_2\sigma_1)\sigma_1$.
In the category we have $\sigma_3\sigma_2 \cdot \sigma_1 \cdot \sigma_3\sigma_2 = \sigma_1\sigma_2 \cdot \sigma_3 \cdot \sigma_2\sigma_1$.

Garside family

category framework

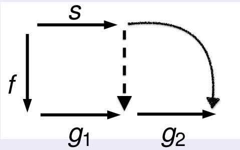
1. We only consider small categories (monoid are categories with one object).
2. We will often the category $\mathscr{C}$ and its set of morphisms $\mathcal{H}om(\mathscr{C})$
3. If x, y are object of a category $\mathcal{C}$, by $\mathscr{C}(x, y)$ we denote the set of morphisms from x to y
4. if f, g are in $\mathcal{H}om(\mathscr{C})$, then fg is defined when the source of g is the target of f.
5. A category is inverse-free if its inverse elements are the unities only.
6. a (sub)-family of a category $\mathscr{C}$ is a subset of $\mathcal{H}om(\mathscr{C})$ equiped with the object set, the source map and the target maps.
7. If S is a family of a category $\mathscr{C}$, a S-path of length q is a sequence $g_1|\cdots|g_q$ of elements of S so that the product $g_1\cdots g_q$ is defined.

Garside family

Definition (**greedy path)**

Assume that $\mathscr{C}$ is a left-cancellative category and S is a family of $\mathscr{C}$. A length two $\mathscr{C}$-path $g_1|g_2$ is called *S-greedy* if

$$\text{each relation } h \preccurlyeq fg_1g_2 \text{ with } h \in S \text{ implies } h \preccurlyeq fg_1.$$



A $\mathscr{C}$-path $g_1|\cdots|g_q$ is called *S-greedy* if $g_k|g_{k+1}$ is S-greedy for each $k < q$.

By definition a length 1 path is greedy.

Garside family

Definition ($\mathcal{S}$-normal path/decomposition)

Let $\mathscr{C}$ be left-cancellative inverse free category $\mathscr{C}$.
Assume $\mathcal{S}$ is a family of $\mathscr{C}$ that contains all the identies.

1. A $\mathscr{C}$-path is $\mathcal{S}$-normal if it is $\mathcal{S}$-greedy and its terms lie in $\mathcal{S}$.
2. A family $\mathcal{S}$ of an inverse free left-cancellative category $\mathscr{C}$ is said to be a *Garside* family in $\mathscr{C}$ if every element of $\mathscr{C}$ admits at least one $\mathcal{S}$-normal decomposition.

Example

$\mathscr{C}$ is a Garside family of $\mathscr{C}$.

Example

Let $n \geqslant 1$ and $L_n = \langle \mathtt{a}, \mathtt{b} \mid \mathtt{a}\mathtt{b}^n = \mathtt{b}^{n+1} \rangle^+$. Set $\mathcal{S}_n = \{1, \mathtt{a}, \mathtt{b}, \mathtt{b}^2, \ldots, \mathtt{b}^{n+1}\}$. Then L_n is left cancellative and $\mathcal{S}_n$ is a Garside family.