Garside Categories

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Amiens 2022 - GDR Tresses

Lecture 2 : Working with Garside families in categories and groupoid.

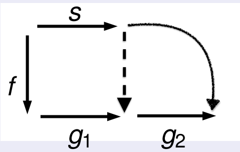
Garside family

Greediness

Definition (**greedy path)**

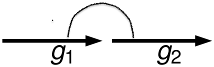
Assume that $\mathscr{C}$ is a left-cancellative category and S is a family of $\mathscr{C}$. A length two $\mathscr{C}$-path $g_1|g_2$ is called *S-greedy* if

$$\text{each relation } h \preccurlyeq fg_1g_2 \text{ with } h \in S \text{ implies } h \preccurlyeq fg_1.$$



A $\mathscr{C}$-path $g_1|\cdots|g_q$ is called *S-greedy* if $g_k|g_{k+1}$ is S-greedy for each $k < q$.

By definition a length 1 path is greedy.

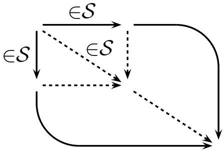
Notation for greediness : 

Garside family

Greediness

Proposition

Assume that $\mathscr{C}$ is a inverse-free left-cancellative category.
Let $\mathcal{S}$ be a family of $\mathscr{C}$ that generates $\mathscr{C}$, contains the identity elements, is closed under right-comultiple is closed under right-divisor.
Then, for every $\mathscr{C}$-path $g_1|g_2$ with g_1 in $\mathcal{S}$, the following are equivalent :
(i) *The path $g_1|g_2$is $\mathcal{S}$-greedy ($h \preccurlyeq fg_1g_2$ and $h \in S \Rightarrow h \preccurlyeq fg_1$) ;*
(ii) *For every h in $\mathcal{S}$, $h \preccurlyeq g_1g_2 \Rightarrow h \preccurlyeq g_1$;*
(iii) *The unique element h of $\mathscr{C}$ satisfying both $g_1h \in \mathcal{S}$ and $h \preccurlyeq g_2$ is 1_x, where x is the target of g_1 .*



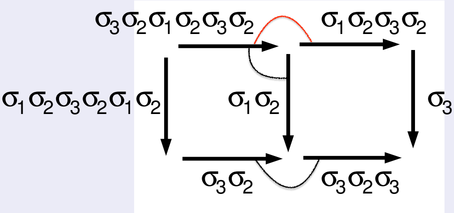
Garside family

Greediness

Example

In B_4 , let $\mathcal{S}$ be the set of left-divisor of Δ. Then, $\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \mid \sigma_1\sigma_2\sigma_3\sigma_2\sigma_3$ and $\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \mid \sigma_1\sigma_2\sigma_3\sigma_2 \mid \sigma_3$ are $\mathcal{S}$ greedy.

(Recall Lecture 1 : $g = \sigma_3{\sigma_2}^2\sigma_1{\sigma_2}^2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$ in B_4^+.)



Example

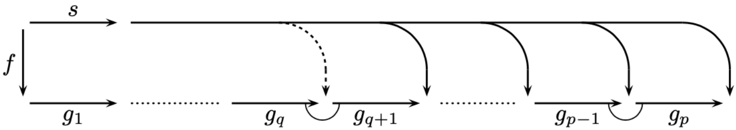
Let $L_2 = \langle \mathrm{a}, \mathrm{b} \mid \mathrm{ab}^2 = \mathrm{b}^3 \rangle^+$. Set $\mathcal{S}_n = \{1, \mathrm{a}, \mathrm{b}, \mathrm{b}^2, \mathrm{b}^3\}$. Then $b \mid a \mid b^2$ is not $\mathcal{S}$ greedy because $a \mid b^2$ is not.

Proposition (**grouping entries)**

Assume that $\mathscr{C}$ is a left-cancellative category and $\mathcal{S}$ is included in $\mathscr{C}$. If a path $g_1 | \cdots | g_q$ is $\mathcal{S}$-greedy, then so is every path obtained from $g_1 | \cdots | g_p$ by replacing adjacent entries by their product. In particular, $g_1 | g_2 \cdots g_p$ is $\mathcal{S}$-greedy, that is

$$\text{each relation } h \preccurlyeq fg_1 \cdots g_q \text{ with } h \in \mathcal{S} \text{ implies } h \preccurlyeq fg_1. \tag{1}$$

proof : $g_1 \cdots g_q \mid g_{p+1} \cdots g_q$ is $\mathcal{S}$-greedy.



Garside family

greediness

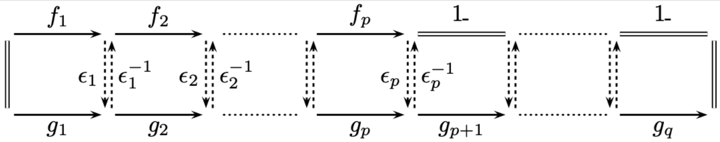
Definition ($\mathcal{S}$-normal path/decomposition)

Assume $\mathcal{S}$ is a family of a left-cancellative inverse free category $\mathscr{C}$.

1. A $\mathscr{C}$-path is $\mathcal{S}$-normal if it is $\mathcal{S}$-greedy and its not identity terms lie in $\mathcal{S}$.
2. A $\mathcal{S}$-normal $\mathscr{C}$-path is strict when no terms is an identity of $\mathscr{C}$. In particular, this is a $\mathcal{S}$-path.
3. A $\mathcal{S}$-normal decomposition of g in $\mathscr{C}$ is a $\mathcal{S}$-normal path $g_1|\cdots|g_p$ so that $g = g_1 \cdots g_p$.

Proposition

For any family $\mathcal{S}$ of an inverse-free left-cancellative category $\mathscr{C}$, every element of $\mathscr{C}$ admits at most one strict $\mathcal{S}$-normal decomposition.



Garside family

Definition (**Garside family)**

A family $\mathcal{S}$ of an inverse free left-cancellative category $\mathscr{C}$ is said to be a *Garside* family in $\mathscr{C}$ if every element of $\mathscr{C}$ admits at least one $\mathcal{S}$-normal decomposition.

In this case, for g in $\mathscr{C}$, by $\|g\|_{\mathcal{S}}$ we denote the number not unity terms of any of its $\mathcal{S}$-normal decomposition. For a identity, this is 0 and for a not identity element this is the number of terms of its strict $\mathcal{S}$-normal decomposition.

Remark

1. *This is equivalent to ask that every not identity element of $\mathscr{C}$ admits a (unique) strict $\mathcal{S}$-normal decomposition.*
2. *If $\mathcal{S}$ is a Garside family of $\mathscr{C}$, then so is $\mathcal{S} \cup \mathcal{S}^2 \cup \cdots \mathcal{S}^k$ for every $k \geq 1$.*

Example

$\mathscr{C}$ is a Garside family of $\mathscr{C}$.

Garside monoid

Definition

A *Garside monoid* is a pair (M, Δ) where M is a monoid such that
• M is left- and right-cancellative,
• there exists $\lambda : M \to \mathbb{N}$ satisfying $\lambda(fg) \geqslant \lambda(f) + \lambda(g)$ and $g \neq 1 \Rightarrow \lambda(g) \neq 0$,
• any two elements of M have a left- and a right-lcm and a left- and a right-gcd,
• Δ is a *Garside element* of M, this meaning that the left- and right-divisors of Δ coincide, generate M, and are finite in number.

Proposition

Let A^+ be any Artin-Tits monoids and ι be the canonical set section from the associated Coxeter group W to A^+. Then

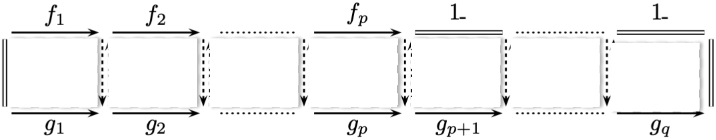
1. *$\iota(W)$ is a Garside family of A^+.*
2. *A^+ is a Garside monoid when W is finite and in this case, $\iota(\omega_0)$ is a Garside element of A^+.*

Garside family

Normal decomposition

Proposition

Assume $\mathcal{S}$ is a family of an inverse free left-cancellative category $\mathscr{C}$. If $g \in \mathcal{S}^p$ possesses a strict $\mathcal{S}$-normal decompostion $g_1|\cdots|g_q$, then $q \leq p$.



Proposition

Let $\mathcal{S}$ is a subfamily of an inverse free left-cancellative category $\mathscr{C}$. Assume that $\mathcal{S}$ generates $\mathscr{C}$. Then the two properties are equivalent

1. *$\mathcal{S}$ is a Garside family*
2. *Any element of $\mathcal{S}^2$ possesses $\mathcal{S}$-normal decomposition.*

Garside family

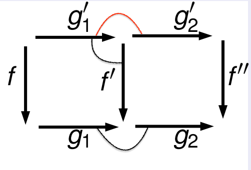
First Domino Rule

Proposition

Assume $\mathscr{C}$ is an inverse free left-cancellative category, $\mathcal{S}$ is a family of $\mathscr{C}$. Assume the $\mathscr{C}$-path $g_1 \mid g_2$ is $\mathcal{S}$-greedy. Let g_1' be a left-multiple of g_1 and g_2' is a left-divisor of g_2. Then $g_1 \mid g_2$ is $\mathcal{S}$-greedy as well.

Proposition (**First domino rule)**

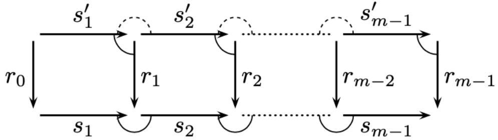
Assume $\mathscr{C}$ is an inverse free left-cancellative category, $\mathcal{S}$ is a family of $\mathscr{C}$, and we have a the following commutative diagram with edges in $\mathscr{C}$.

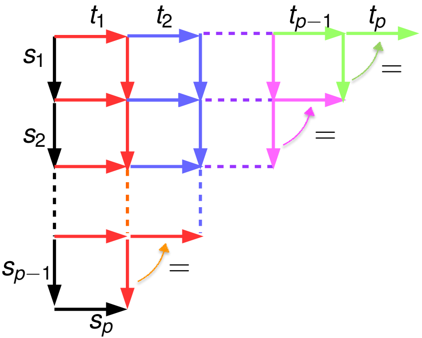


If $g_1|g_2$ and $g_1'|f'$ are $\mathcal{S}$-greedy, then $g_1'|f'g_2$ and $g_1'|g_2'$ are $\mathcal{S}$-greedy as well.

Garside family

proof of the proposition





Transform the $\mathcal{S}$-path $s_1|\cdots|s_p$ into the $\mathcal{S}$-normal path $t_1|\cdots|t_p$.
Each horizontal path is a $\mathcal{S}$-normal path. This takes $p(p-1)/2$ steps.

Introduction and motivations

Braid groups - Normal form

Example

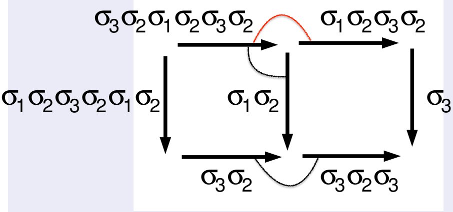
Consider the element $g = \sigma_3\sigma_2{}^2\sigma_1\sigma_2{}^2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$ *in* B_4^+.

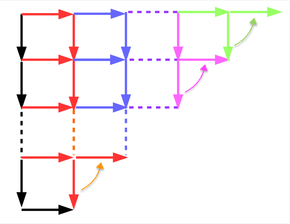
$g = \sigma_3\sigma_2 \cdot \sigma_2\sigma_1\sigma_2 \cdot \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1$

$g = \sigma_3\sigma_2 \cdot \sigma_2\sigma_1\sigma_2 \cdot \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2 \cdot \sigma_2\sigma_1\sigma_2\sigma_3\sigma_2\sigma_1 \cdot \sigma_2\sigma_3\sigma_2 =$

$\sigma_3\sigma_2 \cdot \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1\sigma_2 \cdot \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_1\sigma_2 \cdot \sigma_2\sigma_3\sigma_2 =$

$\sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_1\sigma_2 \cdot \sigma_3\sigma_2\sigma_3 = \sigma_3\sigma_2\sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_1\sigma_2\sigma_3\sigma_2 \cdot \sigma_3 =$





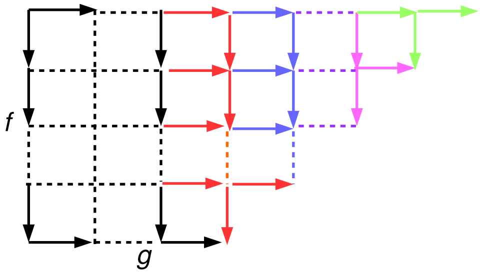
Garside family

Length

Proposition

Assume that $\mathscr{C}$ is a left-cancellative category and $\mathcal{S}$ is a Garside family of $\mathscr{C}$. Then, for every $\mathscr{C}$-path $f|g$, we have

$$\|g\|_{\mathcal{S}} \leqslant \|fg\|_{\mathcal{S}} \leqslant \|f\|_{\mathcal{S}} + \|g\|_{\mathcal{S}}.$$



Garside family

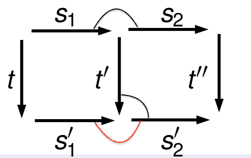
Length

Example

Let $n \geqslant 1$ and $L_n = \langle \mathrm{a}, \mathrm{b} \mid \mathrm{ab}^n = \mathrm{b}^{n+1} \rangle^+$. Set $\mathcal{S} = \{1, \mathrm{a}, \mathrm{b}, \mathrm{b}^2, \dots, \mathrm{b}^{n+1}\}$. Then L_n is left cancellative and S_n is a Garside family (exercice). The strict $\mathcal{S}$-normal decomposition of ab^{n-1} is $a \mid b^{n-1}$, but the $\mathcal{S}$-normal decomposition of $ab^{n-1} \cdot b$ is b^{n+1}. so $\|ab^{n-1} \cdot b\|_{\mathcal{S}} < \|ab^{n-1}\|_{\mathcal{S}}$.

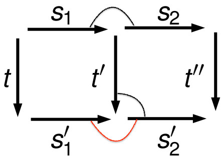
Definition (**Second domino rule)**

Assume $\mathscr{C}$ is a inverse-free left-cancellative category and $\mathcal{S}$ a family in $\mathscr{C}$. The *second domino rule* is valid if, whenever $s_1 | s_2$ and $t' | s_2'$ are $\mathcal{S}$-greedy in a commutative diagram as below with edges in $\mathcal{S}$, then $s_1' | s_2'$ is $\mathcal{S}$-greedy as well.

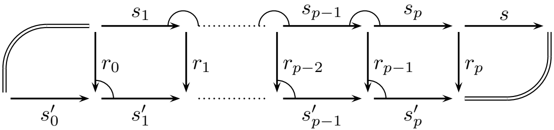


Garside family

Length



Normal decomposition using the second domino rule :



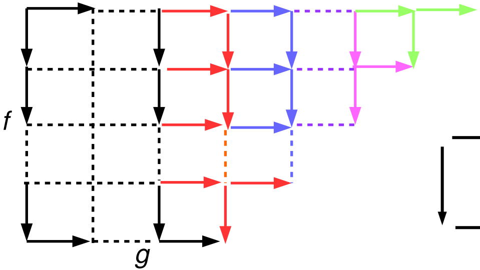
Garside family

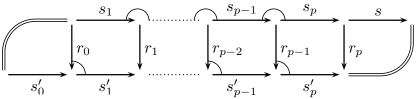
Length

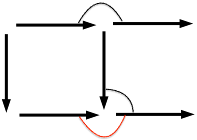
Proposition

Assume that $\mathscr{C}$ is a inverse free left-cancellative category and $\mathcal{S}$ is a Garside family of $\mathscr{C}$ for which the second domino rule is valid and, moreover, every left-divisor of an element of $\mathcal{S}$ lies in $\mathcal{S}$. Then, for all $\mathscr{C}$-path

$$\max(\|f\|_{\mathcal{S}}, \|g\|_{\mathcal{S}}) \leq \|fg\|_{\mathcal{S}} \leq \|f\|_{\mathcal{S}} + \|g\|_{\mathcal{S}}$$







Garside family

Domino rules 1 and 2

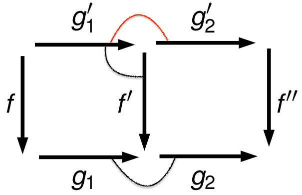


FIGURE – Domino Rule 1

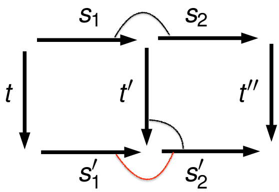


FIGURE – Domino Rule 2

Remark

The equality

$$\max(\|f\|_{\mathcal{S}}, \|g\|_{\mathcal{S}} \leq \|fg\|_{\mathcal{S}} \leq \|f\|_{\mathcal{S}} + \|g\|_{\mathcal{S}}$$

may hold even if the second domino rule does not hold. This is for instance the case when $\mathscr{C}$ is cancellative and $\mathcal{S}$ is both a left Garside family and a right Garside family.

Garside family

Length

Definition (**closure**)

Assume that $\mathscr{C}$ is an inverse free cancellative category and $\mathcal{S}$ is a family in $\mathscr{C}$.
(i) **(left-divisor)** We say that $\mathcal{S}$ is *closed under left-divisor* if every left-divisor of an element of $\mathcal{S}$ is an element of $\mathcal{S}$.
(ii) **(left-comultiple)** We say that $\mathcal{S}$ is *closed under left-comultiple* if every common left-multiple of two elements f, g of $\mathcal{S}$ (if any) is a left-multiple of some common left-multiple of f, g that lies in $\mathcal{S}$.

Proposition

Assume that $\mathscr{C}$ is an inverse free cancellative category and $\mathcal{S}$ is a Garside family of $\mathscr{C}$ such that

(a) *$\mathcal{S} \cup 1_{\mathscr{C}}$ is closed under left-divisor ;*

(b) *$\mathcal{S}$ is closed under left-comultiple.*

Then, the second domino rule is valid for $\mathcal{S}$ in $\mathscr{C}$.

Garside family

Closure property

Proposition

Let $\mathcal{S}$ is a subfamily of an inverse free left-cancellative category $\mathscr{C}$. Assume that $\mathcal{S}$ generates $\mathscr{C}$. Then the two following properties are equivalent

1. *$\mathcal{S}$ is a Garside family.*
2. (a) *$\mathcal{S} \cup 1_{\mathscr{C}}$ is closed under right-divisor ;*
   (b) *$\mathcal{S}$ is closed under right-comultiple ;*
   (c) *Any element of $\mathcal{S}^2$ admits a maximal left divisor in $\mathcal{S}$.*

Definition

Assume that $\mathscr{C}$ is an inverse free cancellative category and $\mathcal{S}$ is a Garside family of $\mathscr{C}$. The family $\mathcal{S}$ is called solid when it contains $1_{\mathscr{C}}$ and it is closed under right-divisor.

So, any Garside family that contains $1_{\mathscr{C}}$ is solid. Later on, we will see caracterisation of Garside family under the assumption that the family is solid.

Garside family

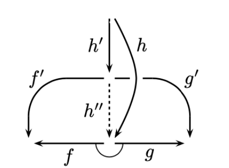
In groupoid

We turn now to groupoid. Here Öre condition will be needed.

Definition (**left-disjoint**)

Two elements f, g of a left-cancellative category $\mathscr{C}$ are called *left-disjoint* if they have the same source and satisfy :

$$\forall h, h' \in \mathscr{C} \ ((h' \preccurlyeq hf \ \& \ h' \preccurlyeq hg\ ) \Rightarrow h' \preccurlyeq h).$$



Notation :

Garside family

In groupoid

The two following result explains why left-disjoint property is relevant in Garside framwork :

Proposition

Assume that $\mathscr{C}$ is a left-Ore category. Then, two elements f, g of $\mathscr{C}$ with the same source are left-disjoint if and only if, for all f', g' in $\mathscr{C}$ such that $f^{-1}g = f'^{-1}g'$ holds in $\mathcal{E}nv(\mathscr{C})$, there exists h satisfying $f' = hf$ and $g' = hg$.

In particular for an inverse free left-Öre category $\mathscr{C}$, any element of $\mathcal{E}nv(\mathscr{C})$ possesses at most one decomposition as a left-disjoint fraction.

Proposition

Assume that $\mathscr{C}$ is a left-cancellative inverse free category, $\mathcal{S}$ is family in $\mathscr{C}$, and $\mathcal{S}$ generates $\mathscr{C}$. For any $\mathcal{S}$-greedy paths $f_1|\cdots|f_p$ and $g_1|\cdots|g_q$, the following are equivalent :

(i) *f_1 and g_1 are left-disjoint.*

(ii) *$f_1 \cdots f_p$ and $g_1 \cdots g_q$ are left-disjoint ;*

Garside family

In groupoid

Definition (symmetric greedy)

Assume that $\mathscr{C}$ is an inverse free left-cancellative category and $\mathcal{S}$ is a family in $\mathscr{C}$. A signed $\mathscr{C}$-path $\overline{f_p}|\cdots|\overline{f_1}|g_1|\cdots|g_q$ is called symmetric $\mathcal{S}$-greedy if

1. the paths $f_1|\cdots|f_p$ and $g_1|\cdots|g_q$ are $\mathcal{S}$-greedy
2. f_1 and g_1 are left-disjoint.

Definition (symmetric normal)

Assume $\mathscr{C}$ is an inverse free left-cancellative category and $\mathcal{S}$ is a family in $\mathscr{C}$. A signed $\mathscr{C}$-path $\overline{f_p}|\cdots|\overline{f_1}|g_1|\cdots|g_q$ is called symmetric $\mathcal{S}$-normal (*resp*. strictly symmetric $\mathcal{S}$-normal) if

1. the paths $f_1|\cdots|f_p$ and $g_1|\cdots|g_q$ are $\mathcal{S}$-normal (*resp*. strictly $\mathcal{S}$-normal)
2. f_1 and g_1 are left-disjoint.

Garside family

In groupoid

Proposition (symmetric normal unique I)

Assume $\mathscr{C}$ is an inverse free left Öre category, $\mathcal{S}$ is a family in $\mathscr{C}$, and $\mathcal{S}$ generates $\mathscr{C}$. Then any element of $\mathcal{E}nv(\mathscr{C})$ admits at most one strict $\mathcal{S}$-normal decompositions

Proposition (symmetric normal exist)

Assume that $\mathscr{C}$ is an inverse free Öre category and $\mathcal{S}$ is a Garside family in $\mathscr{C}$. Then the following are equivalent :

1. *Every element of $\mathcal{E}nv(\mathscr{C})$ admits a symmetric (strict) $\mathcal{S}$-normal decomposition ;*
2. *The category $\mathscr{C}$ admits left-lcms.*

Garside family

Computing symmetric normal decompostion

Definition (**strong Garside**)

Assume that $\mathscr{C}$ is an inverse free category that admits left-lcms. Then a Garside family $\mathcal{S}$ of $\mathscr{C}$ is called strong if for any s, t in $\mathcal{S} \cup 1_{\mathscr{C}}$ with common target possess a left lcm $s \vee t$ and there exists s', t' in $\mathcal{S} \cup 1_{\mathscr{C}}$ so that $s \vee t = s't = t's$. We say that $\mathcal{S}$ is perfect if in addition left lcm lies in $\mathcal{S}$ too.

Proposition

Assume that $\mathscr{C}$ is an inverse free cancellative category that admits local left-lcms. Then, for every Garside family $\mathcal{S}$ of $\mathscr{C}$, the following are equivalent :
(i) $\mathcal{S}$ *is a strong (resp. perfect) Garside family in* $\mathscr{C}$ *;*
(ii) $\mathcal{S} \cup 1_{\mathscr{C}}$ *is closed under left-complement (resp. this and left-comultiple).*
Moreover, when the above conditions are met, $\mathcal{S} \cup 1_{\mathscr{C}}$ *is closed under left-divisor.*

Garside family

Computing symmetric normal decompostion

Proposition (**strong exists)**

Assume that $\mathscr{C}$ is a left-Ore category. Then the following are equivalent :
(i) *Some Garside family of $\mathscr{C}$ is strong ;*
(ii) *The category $\mathscr{C}$ viewed as a Garside family in itself is strong ;*
(iii) *The category $\mathscr{C}$ admits left-lcms.*

Proposition

Assume that $\mathscr{C}$ is an inverse free cancellative category that admits left-lcms and $\mathcal{S}$ is a strong Garside family in $\mathscr{C}$.
(i) *Every element of $\mathcal{E}nv(\mathscr{C})$ that can be represented by a positive–negative $\mathcal{S}$-path of length ℓ admits a symmetric $\mathcal{S}$-normal decomposition of length at most ℓ.*
(ii) *For every positive–negative $\mathcal{S}$-path, there is an algorithm that returns in quadratic time. the symmetric $\mathcal{S}$-normal decomposition of the element of $\mathcal{E}nv(\mathscr{C})$ represented by the $\mathcal{S}$-path.*

Garside family

Computing symmetric normal decomposition

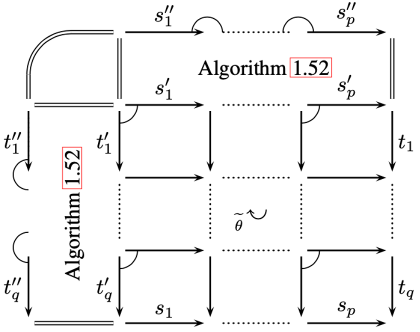


FIGURE – Algo 1

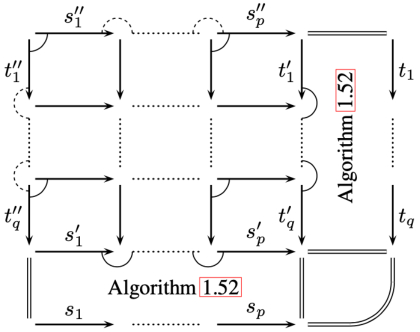


FIGURE – Algo 2

Garside family

Recognizing Garside families

Proposition

Assume that $\mathscr{C}$ is a left-cancellative inverse-free category and $\mathcal{S}$ is a solid generating subfamily of $\mathscr{C}$. Then the following are equivalent :
(i) *The category $\mathscr{C}$ is right-Noetherian and $\mathcal{S}$ is Garside in $\mathscr{C}$;*
(ii) *The family $\mathcal{S}$ is locally right-Noetherian and closed under right-comultiple.*

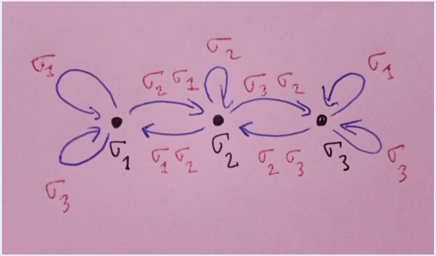
Proposition

Assume that $\mathscr{C}$ is a left-cancellative inverse-free category that is right-Noetherian and admits conditional right-lcms. Then a family $\mathcal{S}$ that generates $\mathscr{C}$ and contains $1_{\mathscr{C}}$ is a Garside family in $\mathscr{C}$ if and only if any one of the following conditions is satisfied :
(i) *$\mathcal{S}$ is closed under right-lcm and right-divisor ;*
(ii) *$\mathcal{S}$ is closed under right-lcm and right-complement ;*

Proposition (**smallest Garside**)

Assume that $\mathscr{C}$ is a strongly-noetherian left-cancellative inverse-free category. Then there exists a smallest Garside family $\mathcal{S}$ in $\mathscr{C}$ containing $1_{\mathscr{C}}$, namely the closure of the atoms under right-mcm and right-divisor.

Example



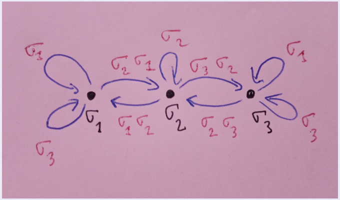
The category of positive quasi-centralisers of generators in B_4 is cancellative inverse-free and strongly-noetherian. Its atom set consists on the red morphisms. So its closure under right-mcm (indeed right lcm) and right-divisor is a Garside family. The second domino rule holds.

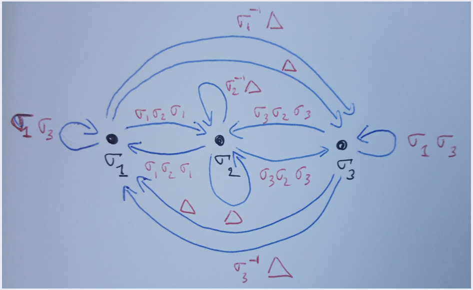
Garside family

Recognizing Garside families

Example

The smallest strong Garside family of the category of positive quasi-centralisers of generators in B_4 is the union of all represented morphisms plus the 3 identity elements.





$$\sigma_2{}^{-1}\Delta_4 = \sigma_3\sigma_2 \cdot \sigma_1 \cdot \sigma_2\sigma_3 = \sigma_1\sigma_2 \cdot \sigma_3 \cdot \sigma_2\sigma_1.$$

Garside family

Presentation

Proposition

Assume that $\mathscr{C}$ is a left-cancellative inverse-free category and $\mathcal{S}$ is a solid Garside family in $\mathscr{C}$. Then $\mathscr{C}$ admits the presentation $\langle \mathcal{S} \mid R \rangle^+$ where R consists on the family of all relations $fg = h$ with f, g, h in $\mathcal{S}$ that are valid in $\mathscr{C}$ make a presentation of $\mathscr{C}$ in terms of $\mathcal{S}$.

Remark

Obviously this presentation is not minimal in terms of generators (since we only need the atoms to generate the category). If S is finite, we get a finite presentation.

Garside family

Presentation

Example

The category $\mathscr{C}$ of positive quasi-centralisers of the generators in B_4 has a presentation with the already seen atom set and the defining relations :

1. *In* $\mathscr{C}(\sigma_1, \cdot)$
   - (a) $\sigma_1\sigma_3 = \sigma_3\sigma_1$;
   - (b) $\sigma_1 \cdot \sigma_2\sigma_1 = \sigma_2\sigma_1 \cdot \sigma_2$;
   - (c) $\sigma_2\sigma_1 \cdot \sigma_3\sigma_2 \cdot \sigma_1 = \sigma_3 \cdot \sigma_2\sigma_1 \cdot \sigma_3\sigma_2$
2. *In* $\mathscr{C}(\sigma_3, \cdot)$:
   - (a) $\sigma_1\sigma_3 = \sigma_3\sigma_1$;
   - (b) $\sigma_3 \cdot \sigma_2\sigma_3 = \sigma_2\sigma_3 \cdot \sigma_2$;
   - (c) $\sigma_2\sigma_3 \cdot \sigma_1\sigma_2 \cdot \sigma_3 = \sigma_1 \cdot \sigma_2\sigma_3 \cdot \sigma_1\sigma_2$
3. *In* $\mathscr{C}(\sigma_2, \cdot)$:
   - (a) $\sigma_2 \cdot \sigma_1\sigma_2 = \sigma_1\sigma_2 \cdot \sigma_1$;
   - (b) $\sigma_2 \cdot \sigma_3\sigma_2 = \sigma_2\sigma_3 \cdot \sigma_3$;
   - (c) $\sigma_3\sigma_2 \cdot \sigma_1 \cdot \sigma_2\sigma_3 = \sigma_1\sigma_2 \cdot \sigma_3 \cdot \sigma_2\sigma_1$