Garside Categories

E. Godelle

Amiens 2022 - GDR Tresses

Lecture 3 :
Existence of Garside families -
Quasi-Garside category.

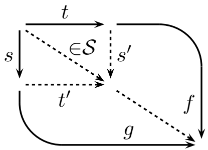
Garside family

subcategory

Proposition

Assume that $\mathscr{C}$ is a left-cancelative, right-noetherian inverse-free category. Assume moreover that $\mathscr{C}$ admits conditional right lcms. Let $\mathcal{S}_1$ be a family of $\mathscr{C}$ containing $1_{\mathscr{C}}$ that is closed under right-diamond. Denote by $\mathscr{C}_1$ the subcategory of $\mathscr{C}$ generated by $\mathcal{S}_1$. Then

1. *$\mathscr{C}_1$ is a closed under right-diamond and closed under right-quotient, is right-noetherian and admits conditional right lcms.*
2. *$\mathcal{S}_1$ is a Garside family of $\mathscr{C}_1$.*



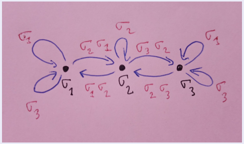
Garside family

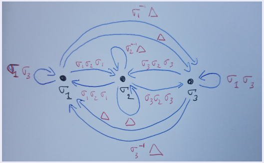
subcategory

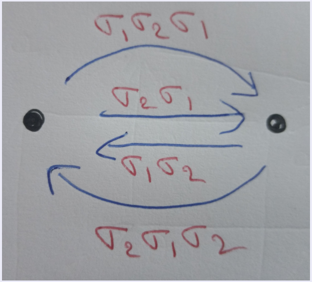
Example

Consider B_3 embeds in B_4 Let $\mathcal{S}$ be the smallest strong Garside family of the category of positive quasi-centralisers of generators in B_4. Then its intersection with the category of positive quasi-centralisers of generators in B_3 is a Garside family.

In other word the family below is a Garside family of the subcategory that it generates in the category of positive quasi-centralisers of generators in B_4.







Garside family

subcategory

Proposition

Assume that $\mathscr{C}$ is a left-cancellative, inverse-free category and $\mathcal{S}$ is a Garside family of $\mathscr{C}$. Assume φ is an automorphism of the category $\mathscr{C}$ such that $\varphi(\mathcal{S}) = \mathcal{S}$. Then $\mathcal{S} \cap \mathscr{C}^{\varphi}$ is a Garside family of $\mathscr{C}^{\varphi}$.

Example

In B_4^+, consider the automorphism φ that fixes σ_2 and exchanges the generators σ_1 and σ_3. Then, it generated by σ_2 and $\sigma_1\sigma_3$. Moreover

$$\{1; \sigma_1\sigma_3, ; \sigma_2; \sigma_1\sigma_3\sigma_2; \sigma_2\sigma_1\sigma_3; \sigma_2\sigma_1\sigma_3\sigma_2; \sigma_1\sigma_3, ; \sigma_2; \sigma_1\sigma_3; \Delta\}$$

is a Garside family of $(B_4^+)^{\varphi}$. The later has the following presentation : $\langle a, b \mid abab = baba \rangle$ with $a = \sigma_1\sigma_3$ and $b = \sigma_2$

Garside family

subcategory

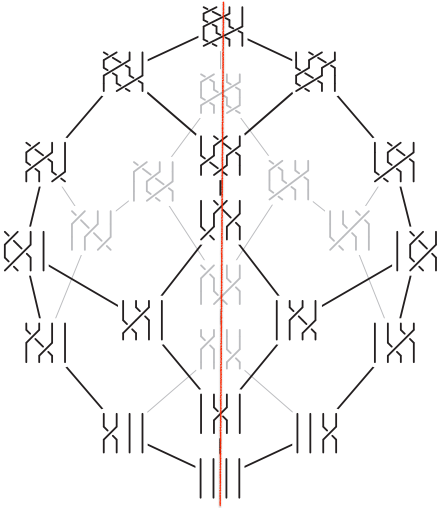


FIGURE – The lattice $(\mathrm{Div}(\Delta_4), \preceq) \cap (B_4^+)^{\varphi}$ with its 8 elements.

Garside family

Parabolic subcategory

Definition

Assume that $\mathscr{C}$ is an inverse-free left cancellative and right-noetherian category. A subcategory $\mathscr{C}_1$ of $\mathscr{C}$ is a parabolic subcategory of $\mathscr{C}$ when it is closed by right-comultiple and by factor.

Proposition

Assume that $\mathscr{C}$ is an inverse-free left cancellative and right-noetherian category. Then

1. *Any parabolic subcategory is strongly compatible with $\mathcal{S}$. That is $\mathcal{S} \cap \mathscr{C}_1$ is a Garside family of $\mathscr{C}_1$ and for any elements of $\mathscr{C}_1$, its $\mathcal{S}$-Normal decomposition ans its $\mathcal{S} \cap \mathscr{C}_1$- normal decomposition coincide.*
2. *any intersection of parabolic subcategories is a parabolic subcategory.*

Working with categories

How to obtain basic properties on categories

Here we are facing some difficulties regarding Garside approach :
In his work Garside prove by induction at the same time that

1. B_n^+ is (left-) cancellative,
2. B_n^+ possesses (left and right) - lcm
3. B_n^+ The normal form exists

But all our results assume, at least that the category is left-cancellative and more properties to obtain properties for the enveloping groupoid.

So we need some technic to obtain some property of a category.

Working with categories

complemented presentation

Definition (**right-complemented**)

A category presentation $(\mathcal{S}, \mathcal{R})$ is said to be *right-complemented*, if $\mathcal{R}$ contains

1. no ε relation,
2. no relation $s\cdots = s\cdots$
3. for each $s \neq t$ in S at most one relation $s\cdots = t\cdots$.

In this case, The (syntactic) right-complement $\theta : \mathcal{S}^2 \to \mathcal{S}^*$ is defined by $\theta(s,s) = \varepsilon$. When $\theta(s,t)$ is defined $s\theta(s,t) = t\theta(t,s)$ holds in $\mathscr{C}$.

Example

The standard presentation of B_n is right-complemented

$$\left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{lll} \sigma_i\sigma_j = \sigma_j\sigma_i & \textit{for} & |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \textit{for} & |i-j| = 1 \end{array} \right\rangle$$

If $|i-j| = 1$, $\theta(\sigma_i, \sigma_j) = \sigma_j$ and for $|i-j| \geqslant 2$ we have $\theta(\sigma_i, \sigma_j) = \sigma_j\sigma_i$.

Working with categories

complemented presentation

Example

1. *The presentation (of monoid)* $\left\langle a, b, c, d, e \;\middle|\; \begin{array}{l} ab = cd \\ aeb = ced \end{array} \right\rangle$ *(of monoid) is not right-complemented*
2. *The presentation* $\left\langle a, b, c \;\middle|\; \begin{array}{l} ab = ba \\ cba = 1 \end{array} \right\rangle$ *is not right-complemented*
3. *The presentation of the category of positive quasi-centralisers of the generators in* B_n *is right-complemented .*
4. *The presentation (of monoid)* $\langle \mathtt{a}, \mathtt{b} \mid \mathtt{a}\mathtt{b}^n = \mathtt{b}^{n+1} \rangle^+$ *is right-complemented but not left-complemented .*

Note that if the category $\mathscr{C}$ possesses a complemented presentation, it has to be inverse-free.

Working with categories

complemented presentation

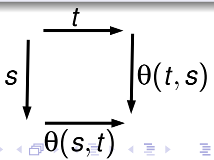
Proposition

Assume that $(\mathcal{S}, \mathcal{R})$ *is a right-complemented presentation. Then the induction rules*

1. $\theta^*(s,s) = \varepsilon_y$ *for* s *in* $\mathcal{S}(-,y)$,
2. $\theta^*(u_1u_2, v) = \theta^*(u_2, \theta^*(u_1, v))$,
3. $\theta^*(u, v_1v_2) = \theta^*(u, v_1)|\theta^*(\theta^*(v_1, u), v_2)$,
4. $\theta^*(\varepsilon_x, u) = u$ *and* $\theta^*(u, \varepsilon_x) = \varepsilon_y$ *for* u *in* $\mathcal{S}^*(x,y)$

define a unique minimal extension θ^* *of* θ^* *into a partial map from* $\mathcal{S}^* \times \mathcal{S}^*$ *to* $\mathcal{S}^*$ *; this map is such that* $\theta^*(u,v)$ *exists if and only if* $\theta^*(v,u)$ *does (in this case the two word represent morphism in the category)*

Here again this is convenient to use a "square" representation.

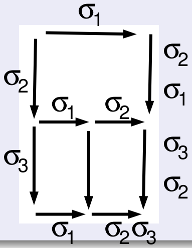


Working with categories

complemented presentation

Example

In B_4, *we have* $\theta^*(\sigma_1, \sigma_2\sigma_3) = \sigma_2\sigma_1\sigma_3\sigma_2$ *and* $\theta^*(\sigma_2\sigma_3, \sigma_1) = \sigma_1\sigma_2\sigma_3$



Working with categories

complemented presentation

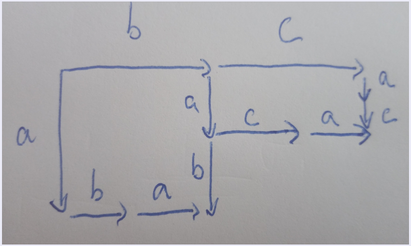
Remark

Even if $\theta : S \times S \to S^*$ *is a map,* $\theta^* : S^* \times S^* \to S^*$ *can be a partial map.*

Example

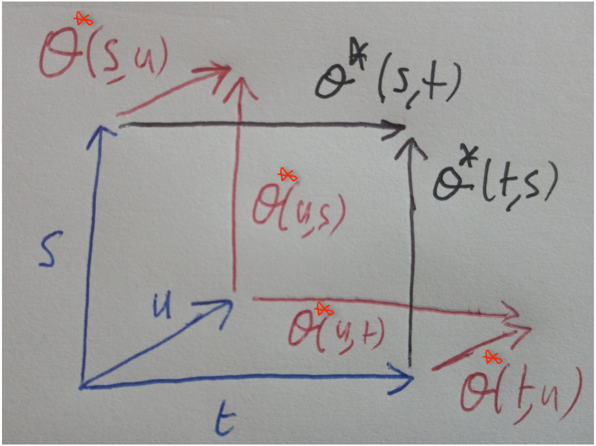
Consider the affine Braid group of type $\tilde{A}_2$*. Its presentation is*

$$\left\langle a, b, c \;\middle|\; \begin{array}{ccc} aba & = & bab \\ cbc & = & bcb \\ aca & = & cac \end{array} \right\rangle$$



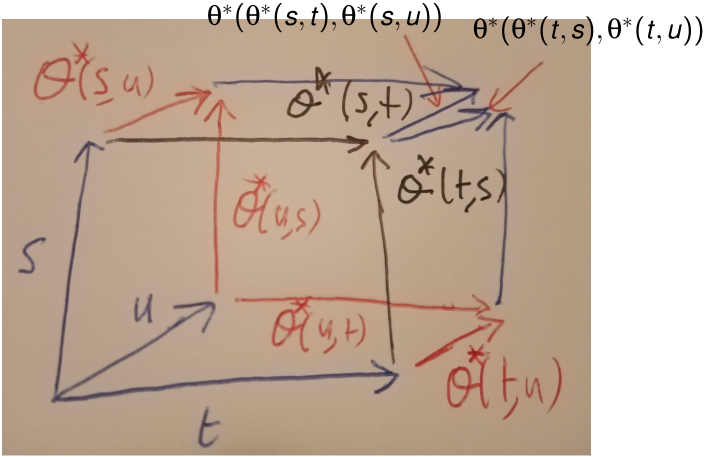
Working with categories

θ cube condition



Working with categories

θ cube condition



Garside family

Presentation

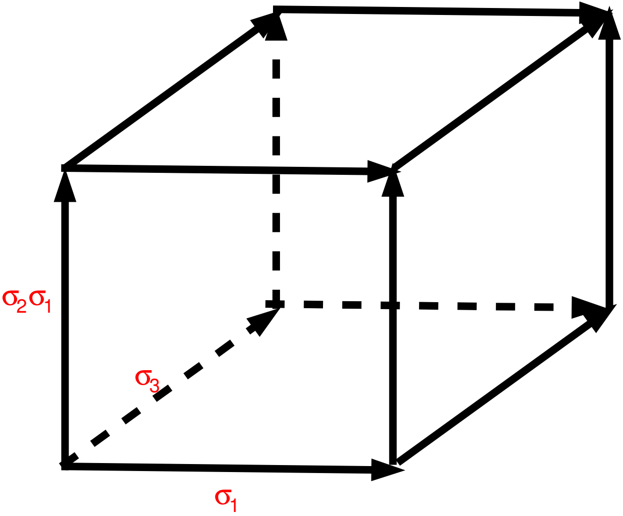
Example

The category $\mathscr{C}$ of positive quasi-centralisers of the generators in B_4 has a presentation with the already seen atom set and the defining relations :

1. *In* $\mathscr{C}(\sigma_1, \cdot)$
   - (a) $\sigma_1\sigma_3 = \sigma_3\sigma_1$;
   - (b) $\sigma_1 \cdot \sigma_2\sigma_1 = \sigma_2\sigma_1 \cdot \sigma_2$;
   - (c) $\sigma_2\sigma_1 \cdot \sigma_3\sigma_2 \cdot \sigma_1 = \sigma_3 \cdot \sigma_2\sigma_1 \cdot \sigma_3\sigma_2$
2. *In* $\mathscr{C}(\sigma_3, \cdot)$ *:*
   - (a) $\sigma_1\sigma_3 = \sigma_3\sigma_1$;
   - (b) $\sigma_3 \cdot \sigma_2\sigma_3 = \sigma_2\sigma_3 \cdot \sigma_2$;
   - (c) $\sigma_2\sigma_3 \cdot \sigma_1\sigma_2 \cdot \sigma_3 = \sigma_1 \cdot \sigma_2\sigma_3 \cdot \sigma_1\sigma_2$
3. *In* $\mathscr{C}(\sigma_2, \cdot)$ *:*
   - (a) $\sigma_2 \cdot \sigma_1\sigma_2 = \sigma_1\sigma_2 \cdot \sigma_1$;
   - (b) $\sigma_2 \cdot \sigma_3\sigma_2 = \sigma_2\sigma_3 \cdot \sigma_3$;
   - (c) $\sigma_3\sigma_2 \cdot \sigma_1 \cdot \sigma_2\sigma_3 = \sigma_1\sigma_2 \cdot \sigma_3 \cdot \sigma_2\sigma_1$

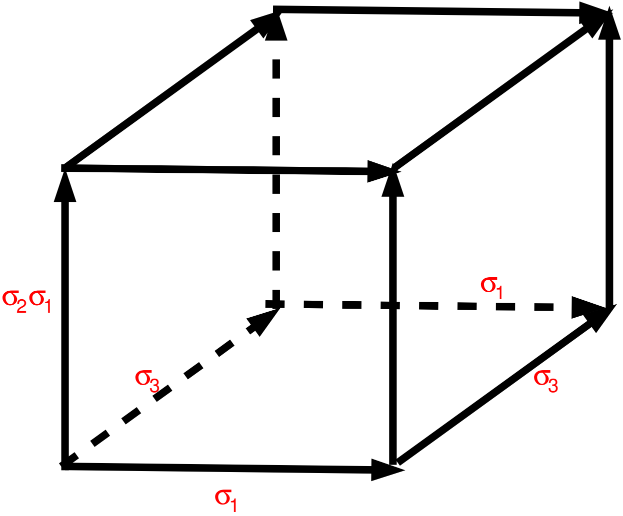
Working with categories

θ cube condition



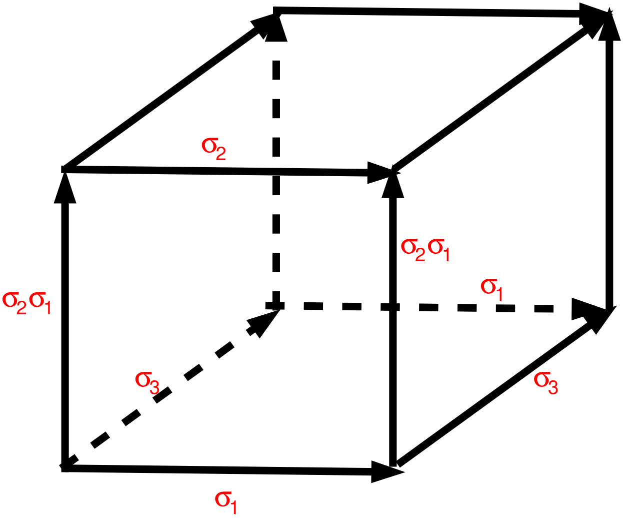
Working with categories

θ cube condition



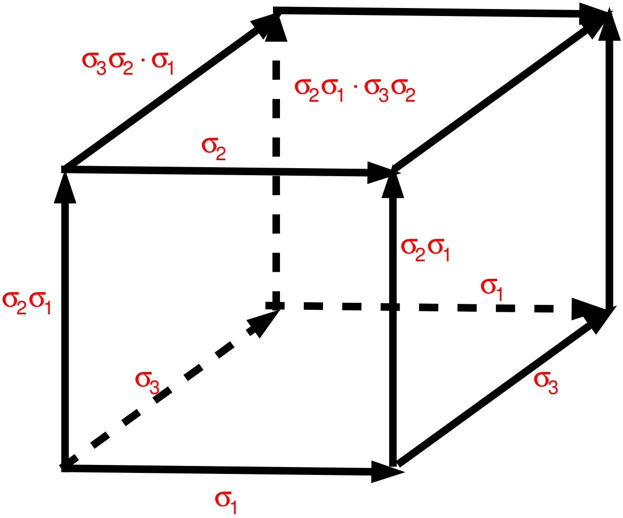
Working with categories

θ cube condition



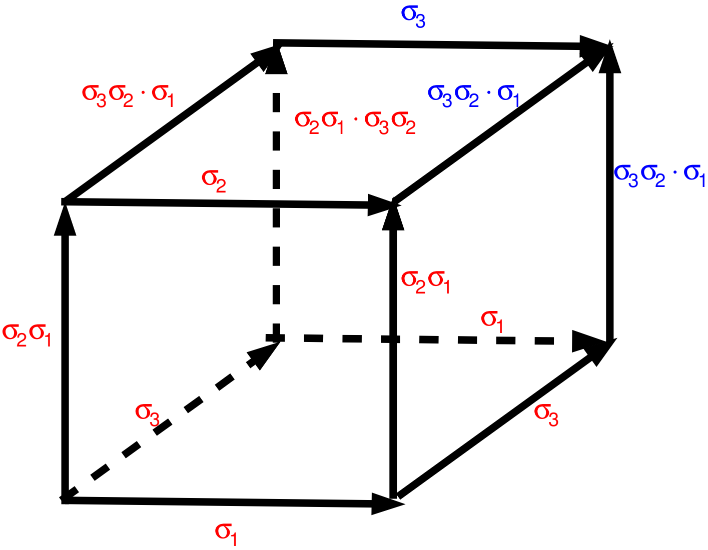
Working with categories

θ cube condition



Working with categories

θ cube condition



Working with categories

θ cube condition

Definition

Assume that $(\mathcal{S}, \mathcal{R})$ is a right-complemented presentation of a category.

1. We say that a triple (u, v, w) of $\mathcal{S}$-paths satisfies the sharp θ-cube condition if either both $\theta^*(\theta^*(s,t), \theta^*(s,u))$ and $\theta^*(\theta^*(t,s), \theta^*(t,u))$ are defined and they are equal, or neither is defined.
2. We say that a triple (u, v, w) of $\mathcal{S}$-paths satisfies the θ-cube condition if either both $\theta^*(\theta^*(s,t), \theta^*(s,u))$ and $\theta^*(\theta^*(t,s), \theta^*(t,u))$ are defined and they are equivalent, or neither is defined.

Working with categories

θ cube condition

Proposition

Assume that $(\mathcal{S},\mathcal{R})$ is a right-complemented presentation of a category $\mathscr{C}$. Assume that one of the following case hold.

1. *$(\mathcal{S},\mathcal{R})$ is short and the sharp θ-cube condition holds for every triple of pairwise distinct elements of $\mathcal{S}$.*
2. *$\mathscr{C}$ is right noetherian and the θ-cube condition holds for every triple of pairwise distinct elements of $\mathcal{S}$.*

Then

1. *$\mathscr{C}$ is left-cancellative.*
2. *$\mathscr{C}$ admits conditional right lcms.*
3. *Any two words u, v in $\mathcal{S}^*$ represent the same element in $\mathscr{C}$ if and only if $\theta^*(u,v)$ and $\theta^*(v,u)$ exists and are empty.*

Working with categories

θ cube condition

Example

The above result can be applied in the following case

1. *The presentation of a Artin-Tits monoids of any type.*
2. *the presentation of the category of positive quasi-centralisers of the generators in B_n.*

Garside germs

The case of Artin groups

Proposition

Let (W,S) be a Coxeter system. Consider the monoid B^+ defined by the following presentation

$$\langle \underline{w}, w \in W \mid \underline{w}\,\underline{w'} = \underline{ww'} \text{ if } \ell_S(ww') = \ell(w) + \ell(w') \rangle$$

Then B^+ is the Artin monoid associated with W and the set $\{underlinew, w \in W\}$ is a Garside family of W.

So we want to extend this situation in the framework of categories.

Garside germs

Definition

A (left-associative) *germ* is a triple $(\mathcal{S}, 1_{\mathcal{S}}, \bullet)$ where

1. $\mathcal{S}$ is a precategory
2. $1_{\mathcal{S}}$ is a subfamily of $\mathcal{S}$ consisting for each object x of an element 1_x with source and target x.
3. $\bullet$ is a partial map from $\mathcal{S}^{[2]}$ into $\mathcal{S}$ such that
   - (a) it respects source and target maps.
   - (b) for all s in $\mathcal{S}(x, y)$ one has $1_x \bullet s = s = s \bullet 1_y$.
   - (c) if $r \bullet s$ and $s \bullet t$ are defined then $(r \bullet s) \bullet t$ is defined if and only if $r \bullet (s \bullet t)$ is defined and in this case they are equal.
   - (d) for any r, s, t in $\mathcal{S}$, if $r \bullet s$ and $(r \bullet s) \bullet t$ are defined then $s \bullet t$ is defined.

Garside germs

Example

Consider the precategory Σ_3 *with one object* $\star$ *defined by* $\Sigma_3 = \{1, \sigma_1, \sigma_2, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_2\sigma_1\sigma_2\}$ *and defined* $s \bullet t$ *by*

$\ulcorner\!\!\rightarrow$	1	σ_1	σ_2	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_2\sigma_1\sigma_2$
1	1	σ_1	σ_2	$\sigma_1\sigma_2$	$\sigma_2\sigma_1$	$\sigma_2\sigma_1\sigma_2$
σ_1	σ_1	—	$\sigma_1\sigma_2$	—	$\sigma_2\sigma_1\sigma_2$	—
σ_2	σ_2	$\sigma_2\sigma_1$	—	$\sigma_2\sigma_1\sigma_2$	—	—
$\sigma_1\sigma_2$	$\sigma_1\sigma_2$	$\sigma_2\sigma_1\sigma_2$	—	—	—	—
$\sigma_2\sigma_1$	$\sigma_2\sigma_1$	—	$\sigma_2\sigma_1\sigma_2$	—	—	—
$\sigma_2\sigma_1\sigma_2$	$\sigma_2\sigma_1\sigma_2$	—	—	—	—	—

Then $(\Sigma_3, 1_\star, \bullet)$ *is an associative germ.*

Garside germs

Definition

Assume $(\mathcal{S}, 1_{\mathcal{S}}, \bullet)$ is an associative germ. The associated category $Cat(\mathcal{S}, 1_{\mathcal{S}}, \bullet)$ is defined by the presentation $\langle \mathcal{S} \mid R \rangle$ where R is the set of relations $fg = h$ such that f, g, h are in $\mathcal{S}$ and $f \bullet g = h$ holds in $\mathcal{S}$.

Example

Consider the associative germ of previous example. Then

$$Cat(\Sigma_3, 1_\star, \bullet) = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$$

Proposition

If $\mathcal{S}$ is a solid Garside family in a inverse-free left cancellative category $\mathscr{C}$, then $\mathcal{S}$ equiped with the induced partial product is an associative germ and the induced category $Cat(\mathcal{S})$ is isomorphic to $\mathscr{C}$.

Garside germs

Definition

Consider a left-associative germ $(\mathcal{S}, 1_{\mathcal{S}}, \bullet)$. For a $\mathcal{S}$-path $s_1 \mid s_2$ in $\mathcal{S}$, set

$$\mathcal{J}(s_1, s_2) = \{t \in \mathcal{S} \mid \exists s, s' \in \mathcal{S} \text{ such that } t = s_1 \bullet s \text{ and } s_2 = s \bullet s'\}$$

A map $I : \mathcal{S} \mid \mathcal{S} \to \mathcal{S}$ is said to be a $\mathcal{J}$-map if the image of any $s_1 \mid s_2$, lies in $\mathcal{J}(s_1, s_2)$.

Definition

A left associative germ $(\mathcal{S}, 1_{\mathcal{S}}, \bullet)$ is a Garside germ if it is left-cancellative and there exists a $\mathcal{J}$-map $I : \mathcal{S} \mid \mathcal{S} \to \mathcal{S}$ such that for any $s_1 \mid s_2 \mid s_3$

$$s_1 \bullet s_2 \text{ is defined } \Rightarrow I(s_1, I(s_2, s_3)) = I(s_1 \bullet s_2, s_3)$$

Proposition

If $(\mathcal{S}, 1_{\mathcal{S}}, \bullet)$ is a Garside germ, then $\mathcal{S}$ is solid Garside family of the left cancellative category $Cat(\mathcal{S})$.

Garside germs

Proposition

A right-Noetherian left-cancellative left-associative germ $\mathcal{S}$ is a Garside germ if and only if $\mathcal{S}$ and satisfies the following property.

For any $s_1 \mid s_2$, any two elements of $\mathcal{J}(s_1, s_2)$ possesses a common right $\mathcal{S}$-multiple that lies in $\mathcal{J}(s_1, s_2)$.

When these conditions are met, the category $\mathcal{C}at(\underline{\mathcal{S}})$ is right-Noetherian.

Proposition

A associative right-Noetherian left-cancellative germ $\mathcal{S}$ that admits right-lcms is a Garside germ.

Bounded Garside families

Definition

A *Garside monoid* is a pair (M, Δ) where M is a monoid such that
- M is left- and right-cancellative,
- there exists $\lambda : M \to \mathbb{N}$ satisfying $\lambda(fg) \geqslant \lambda(f) + \lambda(g)$ and $g \neq 1 \Rightarrow \lambda(g) \neq 0$,
- any two elements of M have a left- and a right-lcm and a left- and a right-gcd,
- Δ is a *Garside element* of M, this meaning that the left- and right-divisors of Δ coincide, generate M, and are finite in number.

In the case of a Garside monoid, the Garside family, and the category possess extra properties. Certainly this situation extends to category context : and lead to the notion of Bounded Garside families.

Bounded Garside families

Definition

Assume that $\mathscr{C}$ is an inverse free left-cancellative category and $\mathcal{S}$ is a Garside family in $\mathscr{C}$ that is closed under left-divisor. We say that $\mathcal{S}$ is right-bounded by a map $\Delta : Obj(\mathscr{C}) \to \mathscr{C}$ when for each object x the morphism $\Delta(x)$ lies in $\mathcal{S}$ and is the common right-multiple of $\mathcal{S}(x, \cdot)$. In this case, we say that Δ is a right-Garside map.

Definition

Assume that $\mathscr{C}$ is an inverse free left-cancellative category and $\mathcal{S}$ is a Garside family in $\mathscr{C}$ that is closed under left-divisor. Then a map $\Delta : Obj(\mathscr{C}) \to \mathscr{C}$ is a right Garside map if and only of the following condition hold

1. $\mathrm{Div}(\Delta)$ generates $\mathscr{C}$
2. $\mathrm{Div}_R(\Delta) \subseteq \mathrm{Div}(\Delta)$
3. for every g in $\mathscr{C}(x, \cdot)$, the elements g and $\Delta(x)$ admit a left gcd.

Bounded Garside families

Example

Let $n \geqslant 1$ and $L_n = \langle \mathtt{a}, \mathtt{b} \,|\, \mathtt{ab}^n = \mathtt{b}^{n+1} \rangle^+$. Set $\mathcal{S} = \{1, \mathtt{a}, \mathtt{b}, \mathtt{b}^2, \ldots, \mathtt{b}^{n+1}\}$. Then $\mathcal{S}$ is right-bounded by $\mathtt{b}^{n+1}$.

Assume $\mathscr{C}$ is an inverse free left-cancellative category and $\mathcal{S}$ is a Garside family in $\mathscr{C}$ that is closed under left-divisor. Assume that a right Garside map $\Delta : Obj(\mathscr{C}) \to \mathscr{C}$ exists.
For s in $\mathcal{S}(x, \cdot)$ we denote by $\partial(s)$ the unique element so that $\Delta(x) = s\partial(s)$.

Note that for every $s \in \mathcal{S}(x, y)$ we have $s\Delta(y) = s\partial(s)\partial^2(s) = \Delta(x)\partial^2(s)$.

Proposition

Under the above assumption there exists a unique functor $\varphi_\Delta : \mathscr{C} \to \mathscr{C}$ that extend ∂^2.

Rem : This functor is not an automorphism in general. For instance in previous example we have $a \cdot b^{n+1} = b^{n+1} \cdot b$ and $b \cdot b^{n+1} = b^{n+1} \cdot b$

Bounded Garside families

Definition

Assume that $\mathscr{C}$ is an inverse free left-cancellative category and $\mathcal{S}$ is a Garside family in $\mathscr{C}$ that is closed under left-divisor. The family $\mathcal{S}$ is said to be bounded by $\Delta : Obj(\mathscr{C}) \to \mathscr{C}$ when

1. Δ is a right Garside map
2. Every s in $\mathcal{S}$ right-divides Δ ($\exists r \in \mathcal{S}(x, \cdot)$ so that $\Delta(x) = rs$).

Proposition

if $\mathscr{C}$ is an inverse free left-cancellative category that possesses a bounded Garside family that it admits common left multiples.

Bounded Garside families

Proposition

Assume that $\mathscr{C}$ is an inverse free left-cancellative category with finite object set and $\mathcal{S}$ is a finite Garside family in $\mathscr{C}$ that is closed under left-divisor.

if $\mathcal{S}$ is right-bounded by a target-injective map Δ then $\mathcal{S}$ is bounded by Δ and φ_Δ is an automorphism.

Proposition

Assume that $\mathscr{C}$ is an inverse free cancellative left-noetherian category and $\mathcal{S}$ is a Garside family closed under left-divisor. and bounded by a target-injective Δ map then

1. *$\mathcal{S}$ is bounded and φ_Δ is an automorphism.*
2. *$Div(\Delta)$ and $Div_R(\Delta)$ coincide.*
3. *$\mathscr{C}$ admits left and right gcds*
4. *$\mathscr{C}$ admits left and right lcms*
5. *$\mathscr{C}$ is a lattice for both right and left divisibility.*