Cactus groups associated with parabolic subgroups

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Cactus groups associated with parabolic subgroups

- Lecture 1 : Why should we care about Cactus groups ?
- Lecture 2 : What can we say about Cactus groups ?

References :

[D] Devadoss S. *Tessellations of moduli spaces and the mosaic operad*. Contemp. Math. 239 (1999), 91-114.

[DJS] Davis, M., Januszkiewicz, T. and Scott, R. *Fundamental groups of blow-ups*. Adv. Math. 177 (2003), 115-179.

[HK] Henriques, A. and Kamnitzer, J. *Crystals and coboundary categories.* Duke Math. 132 (2006), 191-216.

[S] Savage, A. *Braided and coboundary monoidal categories* Contemp. Math., 483 (2009), 229-251.

[B] Bonnafé, C. *Cells and cacti.* Int. Math. Res. Not. (2015) 5775 - 5800.

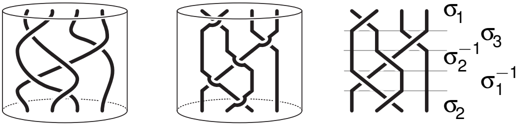
[M] Mostovoy J., *The pure cactus group is residually nilpotent* Arch. Math. 113 (2019) 229-235.

Introduction and motivations

Braid groups

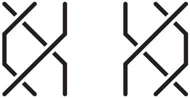
Cactus groups are closed to Braid groups, symmetric groups and RAA(C)G.
The braid group on $n+1$ strands B_{n+1} admits the following presentation

$$\left\langle \sigma_1, \dots, \sigma_n \;\middle|\; \begin{array}{lll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} & |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} & |i-j| = 1 \end{array} \right\rangle \tag{1}$$



$\sigma_1\sigma_3 \equiv \sigma_3\sigma_1$

$\sigma_1\sigma_1^{-1} \equiv \varepsilon \equiv \sigma_1^{-1}\sigma_1$



$\sigma_1\sigma_2\sigma_1 \equiv \sigma_2\sigma_1\sigma_2$

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Symmetric groups

The Symmetric group on $\mathfrak{S}_{n+1}$ admits the following presentation

$$\left\langle s_1, \dots, s_n \;\middle|\; \begin{array}{ccc} s_i s_j = s_j s_i & \text{for} & |i-j| \geqslant 2 \\ s_i s_j s_i = s_j s_i s_j & \text{for} & |i-j| = 1 \\ {s_i}^2 = 1 & & \end{array} \right\rangle \tag{2}$$

We have the exact sequence

$$1 \to PB_{n+1} \to B_{n+1} \xrightarrow{\Upsilon} \mathfrak{S}_{n+1} \to 1$$

Where PB_{n+1} is the pur braid group and σ_i is sent to $\Upsilon(\sigma_i) = s_i$.

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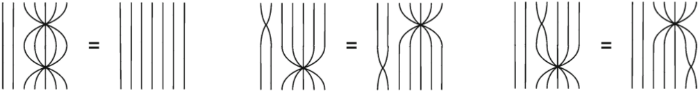
Cactus groups

The Cactus group J_{n+1} admits the following presentation

$$\left\langle \tau_{p,q};\, 1 \leqslant p < q \leqslant n+1 \;\middle|\; \begin{array}{cc} \tau_{p,q}\tau_{m,r} = \tau_{m,r}\tau_{p,q} & \text{for}[p,q]\cap[m,r]=\emptyset \\ \tau_{p,q}\tau_{m,r} = \tau_{m',r'}\tau_{p,q} & \text{for}[m,r]\subset[p,q] \\ \tau_{p,q}^2 = 1 & \end{array} \right\rangle \quad (3)$$

where $m' + r = r' + m = p + q$.

We have a diagram interpretation due to Mostovoy ([M])



We have an exact sequence : $1 \to PJ_{n+1} \to J_{n+1} \xrightarrow{\theta} \mathfrak{S}_{n+1} \to 1$.

Where PJ_{n+1} is the pur cactus group and θ sends $s_{p,q}$ on $\omega_{p,q} = s_p(s_{p+1}s_p)\cdots(s_{q-1}\cdots s_p)$. We have $m' = \omega_{p,q}(r)$ and $r' = \omega_{p,q}(m)$ (one reverses intervals).

Definition

A (strict) monoidal category is a triple $(\mathcal{C}, \otimes, I)$ where $\mathcal{C}$ is a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is an associative bifunctor with an identity object I.

Example

The category of sets can be turned into a monoidal categories using the cartesian product or the disjoint union.

Example

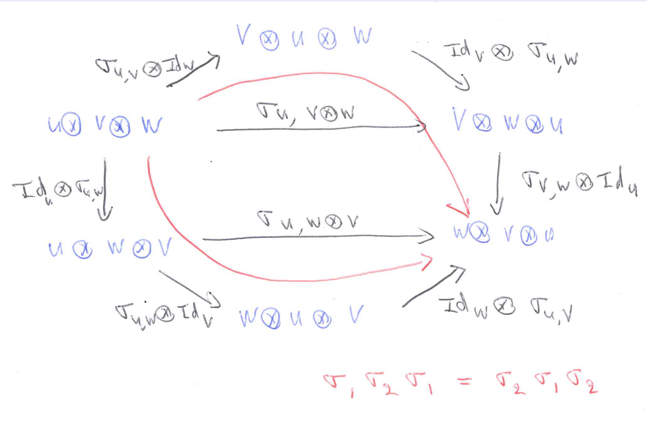
The category of vector spaces on a field $\mathbb{K}$ can be turned into a monoidal category using the direct product.

Cactus groups associated with parabolic subgroups

Cactus groups and coboundary categories

Definition

A monoidal category $(\mathcal{C}, \otimes, I)$ is braided category if for any objects A, B there is a given isomorphism $\sigma_{A,B} : A \otimes B \to B \otimes A$ so that the following is commutative.



$\sigma_1 = \sigma_{.,.} \otimes Id_.$ and $\sigma_2 = Id_. \otimes \sigma_{.,.}$

Cactus groups associated with parabolic subgroups

Cactus groups and coboundary categories

Proposition

Let $(\mathcal{C}, \otimes, I)$ be a braided category, for any objects $A_1, \cdots, A_{n+1}$ of $\mathcal{C}$ and any braid σ in B_{n+1} there is a well defined isomorphism

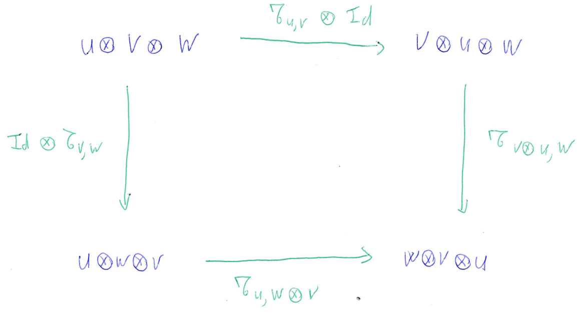
$$\sigma_{A_1,\cdots,A_{n+1}} : A_1 \otimes A_2 \otimes \cdots \otimes A_{n+1} \to A_{\hat{\sigma}(1)} \otimes A_{\hat{\sigma}(2)} \otimes \cdots \otimes A_{\hat{\sigma}(n+1)}$$

with $\hat{\sigma} = \Upsilon(\sigma)$ so that

$$(\sigma\sigma')_{A_1,\cdots,A_{n+1}} = \sigma'_{A_{\hat{\sigma}(1)},\cdots,A_{\hat{\sigma}(n+1)}} \circ \sigma_{A_1,\cdots,A_{n+1}}$$

Definition

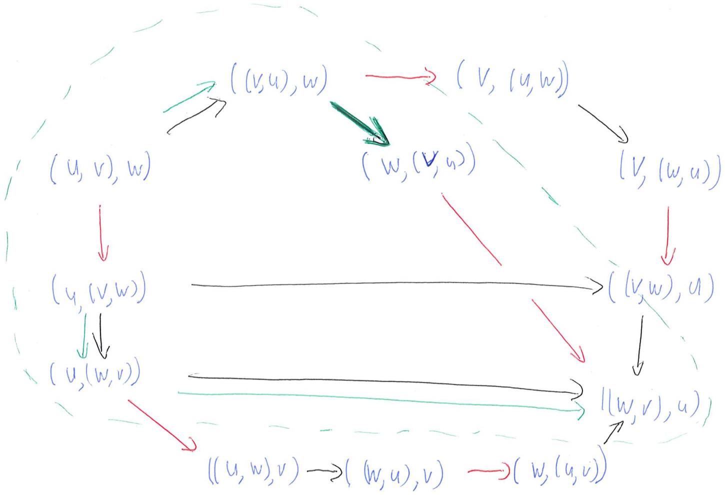
A monoidal category $(\mathcal{C}, \otimes, I)$ is a coboundary category if for any two objects A, B there is a (fixed) isomorphism $\tau_{A,B} : A \otimes B \to B \otimes A$ so that $\tau_{A,B} \circ \tau_{B,A} = Id$ and the following diagram is commutative.



Coboundary categories were introduced by Drinfel'd (1990) in its study of coboundary Hopf algebras. They were associated to Crystal in [HK] .

Cactus groups associated with parabolic subgroups

Cactus groups and coboundary categories



Definition

Consider a coboundary category $(\mathcal{C}, \otimes, I)$ for $1 \leq p < q \leq n+1$ and objects $A_1, \cdots A_{n+1}$, we set

$$\tilde{\tau}_{p,q,(A_1\cdots,A_{n+1})} = Id_{A_1\otimes\cdots\otimes A_{p-1}} \otimes \tau_{A_p,A_{p+1}\otimes\cdots\otimes A_q} \otimes Id_{A_{q+1}\otimes\cdots\otimes A_{n+1}}$$

from $A_1 \otimes \cdots \otimes A_{n+1}$ to
$A_1 \otimes \cdots \otimes A_{p-1} \otimes A_{p+1} \otimes \cdots \otimes A_q \otimes A_p \otimes A_{q+1} \otimes \cdots \otimes A_{n+1}$

So $\tilde{\tau}_{1,2,A,B} = \tau_{A,B} : A \otimes B \to B \otimes A$.

Definition

1. $\tau_{p,p+1,*} = \tilde{\tau}_{p,p+1,*} = Id \otimes \tau_{A_p,A_{p+1}} \otimes Id$.
2. $\tau_{p,q,*} = \hat{\tau}_{p,q,*} \circ \tau_{p+1,q,*}$ for $q > p+1$.

$\tau_{p,q,*} : A_1 \otimes \cdots \otimes A_{n+1} \to$
$A_1 \otimes \cdots \otimes A_{p-1} \otimes A_q \otimes A_{q-1} \otimes \cdots A_p \otimes A_{q+1} \otimes \cdots \otimes A_{n+1}$

Cactus groups associated with parabolic subgroups

Cactus groups and coboundary categories

Proposition ([HK])

we have

$$\begin{aligned} &\tau_{p,q,*}\tau_{m,r,*} = \tau_{m,r,*}\tau_{p,q,*} && \textit{for} \quad [p,q]\cap[m,r]=\emptyset;\\ &\tau_{p,q,*}\tau_{m,r,*} = \tau_{m',r',*}\tau_{p,q,*} && \textit{for} \quad [m,r]\subset[p,q];\\ &\tau^2_{p,q,*} = 1. \end{aligned}$$

Proposition

Let $(\mathcal{C},\otimes,I)$ *be a coboundary category. For any objects* $A_1,\cdots,A_{n+1}$ *of* $\mathcal{C}$ *and any cactus* τ *in* J_{n+1} *there is a well defined (natural) isomorphism*

$$\tau_{A_1,\cdots,A_{n+1}} : A_1\otimes A_2\otimes\cdots\otimes A_{n+1} \to A_{\hat\tau(1)}\otimes A_{\hat\tau(2)}\otimes\cdots\otimes A_{\hat\tau(n+1)}$$

with $\hat\tau = \theta(\tau)\in\mathfrak{S}_{n+1}$ *so that*

$$(\tau\tau')_{A_1,\cdots,A_{n+1}} = \tau'_{A_{\hat\tau(1)},\cdots,A_{\hat\tau(n+1)}}\circ\tau_{A_1,\cdots,A_{n+1}}$$

Cactus groups associated with parabolic subgroups

Cactus groups and configuration space

Definition

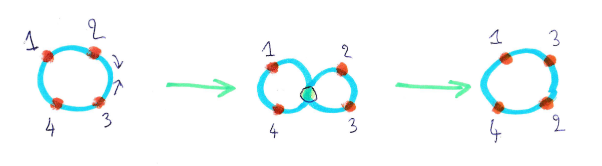
Let X_n be the configuration space of $n+1$ points on the circle, that on $\mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$.

$$X_n = \left((\mathbb{P}^1)^{n+1} \setminus \Delta \right) \Big/ PGL_2(\mathbb{R})$$

Let $M_n = \overline{M}_{0,n+1}(\mathbb{R})$ be the Deligne-Munford compactification of X_n.

Example

In M_3

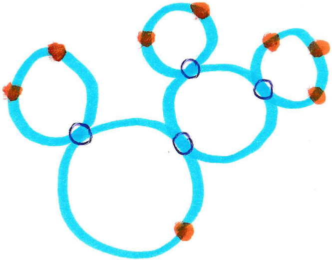


Cactus groups associated with parabolic subgroups

Cactus groups and configuration space

An element of M_n is a circle that is possibly degenerate with a finite set of double points such that :

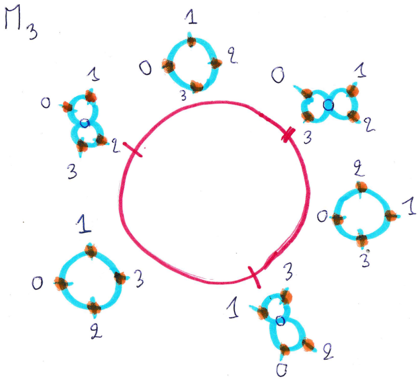
1. double points are distincts from the marked points.
2. the graph of the components is a tree.
3. the automorphism group of the curve is trivial
4. on each component there is at least three points which are either marked or double.



A element of M_7 : a cactus

Cactus groups associated with parabolic subgroups

Cactus groups and configuration space

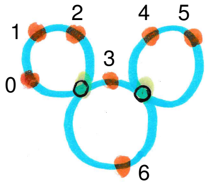


Proposition

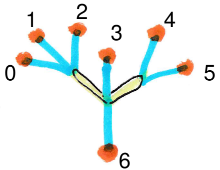
M_n is a smooth compact manifold of dimension $n-2$

Cactus groups associated with parabolic subgroups

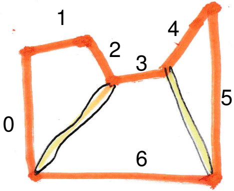
Cactus groups and configuration space



A cactus in M_7



A planar graph in M_7



A polygon with partial diagonal in M_7

(123(4(56)7))

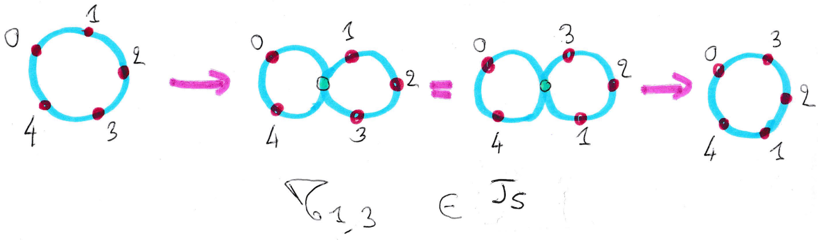
A polygon with partial diagonal in M_7

Cactus groups associated with parabolic subgroups

Cactus groups and configuration space

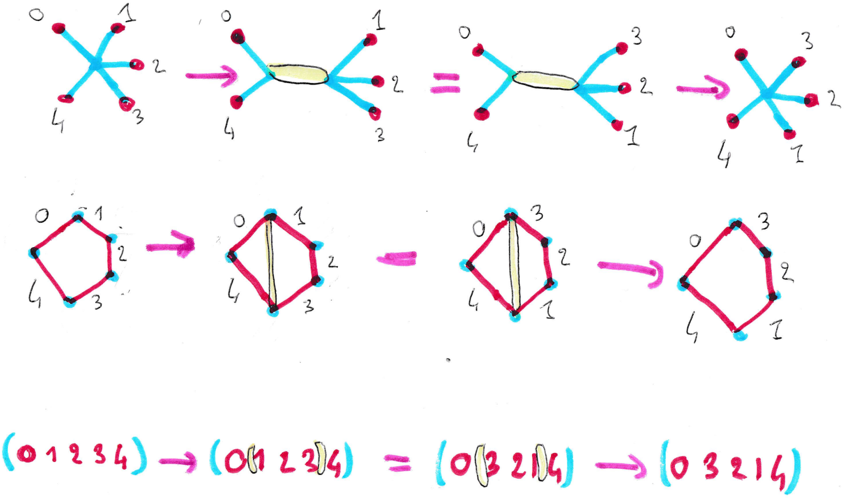
Proposition ([DJS])

1. $\pi_1(M_n) = PJ_n$
2. *for* $i \geq 2$, $\pi_i(M_n) = \{1\}$



Cactus groups associated with parabolic subgroups

Cactus groups and configuration space



Cactus groups associated with parabolic subgroups

Cactus groups and Coxeter groups

Definition

A Coxeter graph Γ is a finite simple labeled graph (V, E, m) where the labeled map m takes her values in $\{3, 4, ...\} \cup \{\infty\}$. The associated Coxeter group W_Γ is defined by the following presentation with generating set V :

$$W = \left\langle V \;\middle|\; \begin{array}{lll} s^2 = 1 & ; & s \in V \\ sts\cdots = tst\ldots & ; & \{s,t\} \in E \text{ and } m(s,t) \neq \infty \end{array} \right\rangle$$

The length of an element of W_Γ is the minima possible length of one of its word representative on V.

Example

The Coxeter group associated with the linear graph with n vertices is the symmetric group $\mathfrak{S}_{n+1}$. The associated Artin group is the braid group B_{n+1}. The length of an element correspond to its number of inversions.

Cactus groups associated with parabolic subgroups

Cactus groups and Coxeter groups

Proposition

When the Coxeter group W_Γ is finite and irreducible (that Γ is connected), then

1. *it possesses a unique element of maximal length ω_V.*
2. *we have $\omega_V V \omega_V^{-1} = V$ and $\omega_V^2 = 1$.*

Example

In the symmetric group $\mathfrak{S}_{n+1}$, we have

1. $\omega_V = s_1(s_2 s_1)\cdots(s_n \cdots s_1)$.
2. $\omega_V s_i = s_{n+1-i}\omega_V$.

Cactus groups associated with parabolic subgroups

Cactus groups and Coxeter groups

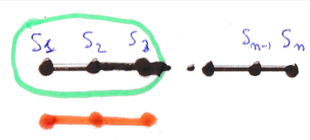
Proposition

Consider the Coxeter group W_Γ associated with $\Gamma = (V, E, m)$.
For $X \subseteq V$, the subgroup W_X of W_Γ is a Coxeter group associated with the full subgraph of Γ spanned by X : the morphism $W_{\Gamma_X} \to W_\Gamma, s \in X \mapsto s \in V$ is into.

Proposition

For $X \subset V$ we have $\omega_V \omega_X \omega_V^{-1} = \omega_Y$ and $\omega_V W_X \omega_V^{-1} = W_Y$ with $\omega_V X \omega_V^{-1} = Y$

Example



Cactus groups associated with parabolic subgroups

Cactus groups and Coxeter groups

Definition

Consider W_Γ associated with $\Gamma = (V, E, m)$. Let F be the set of not empty subsets X of V so that W_X is irreducible and finite. The cactus $C(W_\Gamma)$ is defined by the presentation of group with F for generating set and the defining relatiion :

	(c1)	$c_X^2 = 1$	;	$X \in F$
relatiion :	(c2)	$c_X c_Y = c_{\omega_X(Y)} c_X$	;	$Y \subset X$ and $\omega_X(Y) = \omega_X Y \omega_X^{-1}$
	(c3)	$c_Y c_X = c_X c_Y$	;	$Y \cap X = \emptyset$ and $W_{X \cup Y}$ not irreducible

Example

J_n is the cactus group $C(\mathfrak{S}_n)$ associated with the symmetric group $\mathfrak{S}_n$.

Proposition

For any Coxeter group W, the map $c_X \mapsto \omega_X$ induces an exact sequence

$$1 \to PC(W) \to C(W) \to W \to 1$$

Dual cactus groups

Definition

Let (W,S) be a finite Coxeter system. By T denote its set of reflections. Fix a Coxeter element c.

Then (W,T,c) is a dual Coxeter system. An element δ of W is parabolic relatively to c if $\ell_T(w)+\ell_T(w^{-1}c)=\ell_T(c)$.

Example

in the symmetric group $\mathfrak{S}_{n+1}$, the element $s_1\cdots s_n$ is a Coxeter element.

Dual cactus groups

Proposition

Let $t_1 \cdots t_k = \delta$ *be a decompostion over* T *of a parabolic element* δ *with* $k = \ell_T(\delta)$. *Then*

1. *The subgroup* W_δ *of* W *generated by* $t_1, \cdots, t_k$ *depends on* δ *only.*
2. $(W_\delta, T \cap W_\delta, \delta)$ *is a dual Coxeter system.*
3. $\delta = \delta' \iff W_\delta = W_{\delta'}$.

There is a natural partial order on the set of parabolic elements relatively to c and a notion of irreducible parabolic elements.

Question

1. *What can be said about Cactus groups ?*
2. *How far are they from Coxeter groups ? Artin groups ? RACG ? Garside groups ?*
3. *Is there a notion of dual Cactus groups ?*

$$s_1(s_2s_1)\cdots(s_n\cdots s_1) \to s_n\cdots s_1$$

$$S \to T$$

Answers : next talk

Thanks !