Cactus groups associated with parabolic subgroups

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Cactus groups associated with parabolic subgroups

The objective of today is to

1. Compare Cactus groups with Coxeter/Artin/RACG/Garside groups
2. Introduce the notion of a tricke group.
3. Define a good notion of a dual cactus group.

This talk is mainly based on a joint work with Paolo Bellingeri and Luis Paris (the preprint is expected to be soon in the arxiv).

Cactus groups associated with parabolic subgroups

Definition

The Cactus group J_n has generating set $\{\sigma_{p,q} \mid 1 \leq p < q \leq n\}$ and defining relations

$$\left\{\begin{array}{llll} (1) & \sigma_{p,q}^2 = 1 & ; & \\ (2) & \sigma_{p,q}\sigma_{m,r} = \sigma_{m,r}\sigma_{p,q} & ; & [p,q] \cap [m,r] = \emptyset; \\ (3) & \sigma_{p,q}\,\sigma_{m,r} = \sigma_{p+q-r,p+q-m}\,\sigma_{p,q} & ; & [m,r] \subseteq [p,q]. \end{array}\right.$$

We have an exact sequence : $1 \to PJ_n \to J_n \to \mathfrak{S}_n \to 1$

where $\sigma_{p,q} \mapsto s_{p,q}$.
($s_{p,q}$ exchanges r and $q+p-r$ in $[p;q]$ and fixes the others elements.)

Remark

Not torsion relations are quadratic quasi-commuting relations :
we are closed to RACG.

Cactus groups associated with parabolic subgroups

Definition

Consider a Coxeter system (W, S). For $X \subseteq S$, let ω_X be the longuest element of the subgroup W_X generated by X when W_X is finite. The Cactus group associated with W is generated by the set

$$P_S = \{\underline{\omega}_X \mid X \neq \emptyset \subseteq S; W_X \text{ finite and irreducible}\}$$

$$J(W_S) = \left\langle P_S \;\middle|\; \begin{array}{lll} \underline{\omega}_X^2 = 1 & ; & \\ \underline{\omega}_X\underline{\omega}_Y = \underline{\omega}_Y\underline{\omega}_X & ; & X \cup Y \text{ is not irreducible;} \\ \underline{\omega}_X\underline{\omega}_Y = \underline{\omega_Y^{\omega_X}}\,\underline{\omega}_X & ; & Y \subseteq X \end{array} \right\rangle$$

Proposition

1. *The conjugation $\omega_Y \mapsto \omega_Y^{\omega_X}$ induces a permutations of P_X*
2. *If $X \cup Y$ is not irreducible, then $\omega_X\omega_Y = \omega_Y\omega_X$ in W_S.*

So we have a morphism of groups $p : J(W_S) \to W_S$ defined by $\underline{\omega}_X \mapsto \omega_X$.

Cactus groups associated with parabolic subgroups

Example

How do we solve the word problem in the free group F_n ?

1. read the word letter by letter from left to right
2. cancel if you can with its left neigbor in the obtained word
3. read the final obtained word

Rmk : you solve the word problem but you also get a normal form.

Are $bacbb^{-1}c^{-1}cc^{-1}b$ and $a^{-1}abb^{-1}bac^{-1}ca$ equal in F(a,b,c) ?

$bacbb^{-1}c^{-1}cc^{-1}b \to bacc^{-1}cc^{-1}b \to bab$ and $a^{-1}abb^{-1}bac^{-1}ca \to ba^2$.

The two words does not represent the same element.

Cactus groups associated with parabolic subgroups

Example

How do we solve the word problem in the free Abelian group of rank n ?

1. read the word letter by letter from left to right
2. push on the left as far as possible and cancel if if possible during the process
3. read the final obtain word

Rmk : you solve the word problem but you do not get a normal form.

Are $aacb^{-1}bc^{-1}b$ and $ba^{-1}b^{-1}ac^{-1}baca$ equal in F(a,b,c) ?

$aacb^{-1}bc^{-1}b \to b^{-1}caabc^{-1}b \to ba^2$;
$ba^{-1}b^{-1}ac^{-1}baca \to c^{-1}baca \to a^2b$.

To fix the problem : Choose a total order on the positive letters and push with respect t o the order (for positive and negative letters).

Set $a < b < c$: $aacb^{-1}bc^{-1}b \to aab^{-1}cbc^{-1}b \to a^2b$;
$ba^{-1}b^{-1}ac^{-1}baca \to c^{-1}baca \to bc^{-1}aca \to bc^{-1}caa \to a^2b$.

Cactus groups associated with parabolic subgroups

RAA/CG

Definition

Let $\Gamma = (V, E)$ be a finite simple graph. The RAAG / RACG associated with Γ is the group A_Γ / W_Γ with generating set $\{\sigma_{p,q} \mid 1 \leq p < q \leq n\}$ and defining relations

$$\begin{cases} (1) & \sigma_{p,q}^2 = 1 & ; & \sigma \in V \\ (2) & \sigma\tau = \tau\sigma & ; & \{\sigma, \tau\} \in E; \end{cases}$$

The solution to the word problem : read letters from left to right, push each on the left as far as possible and simplify when possible.

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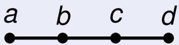

One has $ab = ba$; $bc = cb$ and $dc = cd$.

Is $bacb^{-1}d^{-1}bc^{-1}b$ equal to 1 ?

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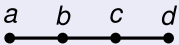

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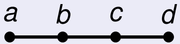

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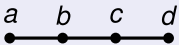

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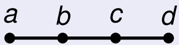

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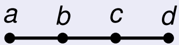

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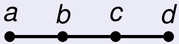

One has $ab = ba$; $bc = cb$ *and* $dc = cd$.

Is $bacb^{-1}d^{-1}bc^{-1}b$ *equal to* 1 ?

$$acc^{-1}d^{-1}b^2$$

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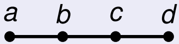

One has $ab = ba$; $bc = cb$ and $dc = cd$.

Is $bacb^{-1}d^{-1}bc^{-1}b$ equal to 1 ?

$$ad^{-1}b^2$$

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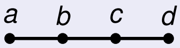

One has $ab = ba$; $bc = cb$ *and* $dc = cd$.

Is $bacb^{-1}d^{-1}bc^{-1}b$ *equal to* 1 ?

$$ad^{-1}b^2 \neq \varepsilon \text{. The answer is : no.}$$

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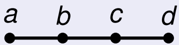

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Is $bacb^{-1}d^{-1}bc^{-1}b$ equal to 1 ?

$$ad^{-1}b^2 \neq \varepsilon\,.\ \textit{The answer is : no.}$$

Remark : Again this does not provide a normal form. ($ad \mapsto da$ and $da \mapsto ad$).

Example

In the group A with group presentation

$$\langle a_1, a_2, a_3, a_4 \mid a_1 a_4 = a_4 a_1 \ ; \ a_2 a_3 = a_3 a_2 \ ; \ a_2 a_4 = a_4 a_2 \rangle$$

we can calculate the piling p of the word $a_2^{-2} a_4^{-1} a_3 a_2 a_4 a_1 a_2 a_1^{-1} a_2^2 a_4^{-1}$ as indicated in Figure 1.

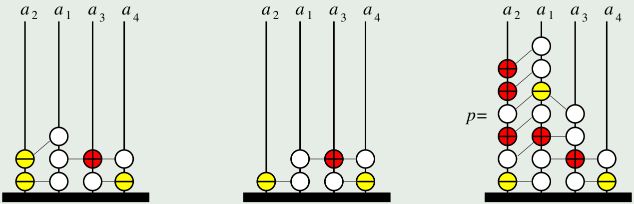

FIGURE – The pilings of the prefixes $a_2^{-2} a_4^{-1} a_3$ and $a_2^{-2} a_4^{-1} a_3 a_2$, and of the full word $a_2^{-2} a_4^{-1} a_3 a_2 a_4 a_1 a_2 a_1^{-1} a_2^2 a_4^{-1}$

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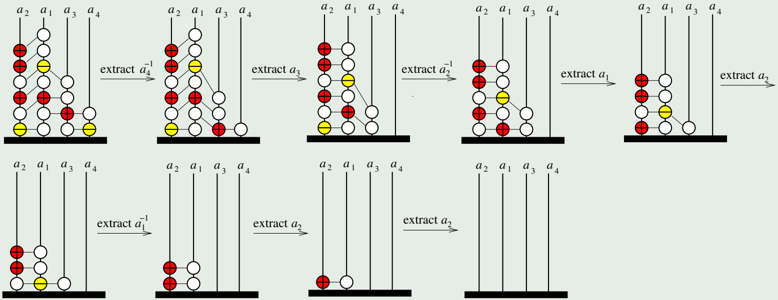
Example

In the group $A = \langle a_1, a_2, a_3, a_4 \mid a_1 a_4 = a_4 a_1 \ ; \ a_2 a_3 = a_3 a_2 \ ; \ a_2 a_4 = a_4 a_2 \rangle$ we have

$$a_2^{-2} a_4^{-1} a_3 a_2 a_4 a_1 a_2 a_1^{-1} a_2^2 a_4^{-1} = a_4^{-1} a_3 a_2^{-1} a_1 a_2 a_1^{-1} a_2 a_2$$



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Consider a Coxeter system (W,S). For $X \subseteq S$, let ω_X be the longuest element of the subgroup W_X generated by X when W_X is finite. The Cactus group associated with W is generated by the set

$$P_S = \{\underline{\omega}_X \mid X \neq \emptyset \subseteq S;\, W_X \text{ finite and irreducible}\}$$

$$J(W_S) = \left\langle P_S \;\middle|\; \begin{array}{lll} \underline{\omega}_X^2 = 1 & ; & \\ \underline{\omega}_X\underline{\omega}_Y = \underline{\omega}_Y\underline{\omega}_X & ; & X \cup Y \text{ is not irreducible;} \\ \underline{\omega}_X\underline{\omega}_Y = \underline{\omega_Y^{\omega_X}}\,\underline{\omega}_X & ; & Y \subseteq X \end{array} \right\rangle$$

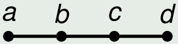
Proposition

1. *The conjugation $\omega_Y \mapsto \omega_Y^{\omega_X}$ induces a permutations of P_X*
2. *If $X \cup Y$ is not irreducible, then $\omega_X\omega_Y = \omega_Y\omega_X$ in W_S.*

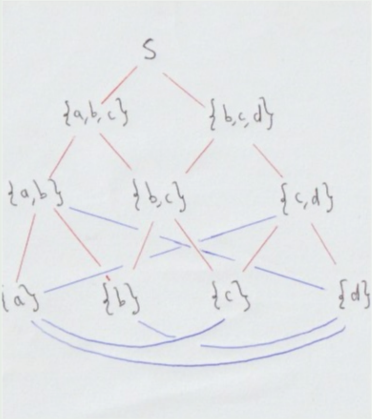
So we have a morphism of groups $p : J(W_S) \to W_S$ defined by $\underline{\omega}_X \mapsto \omega_X$.

Normal form and word problem in Cactus groups

Example

Consider the permutation groups S_5. This is a Coxeter group with Coxeter graph 

and $S = \{a, b, c, d\}$. The set P_S has 10 elements :



There is a natural partial order on P_S given by inclusion (in red) that can be extended to a total order using the alphabetical order. So

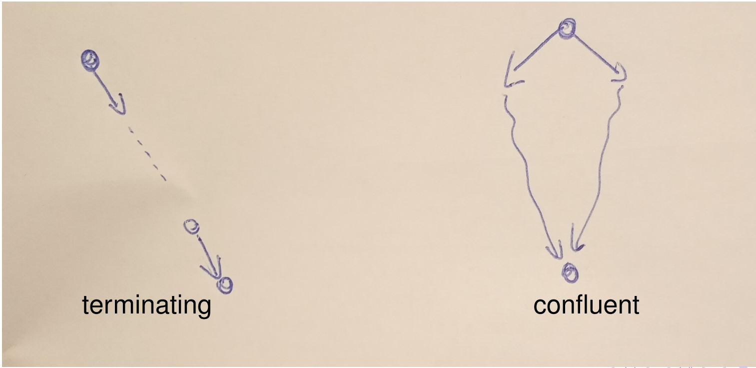
$$\{a\} < \{a, b\} < \{b, c\} < \{b, c, d\}$$

Normal form and word problem in Cactus groups

We want a solution to the word problem.

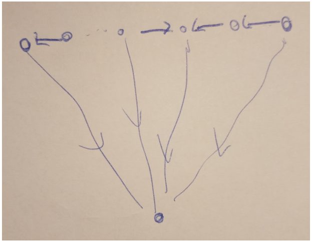
We would like to have a normal form : $\varphi : J(W_S) \to (P_S)^*$ and an algorithm that associated to any word over $P(S)$ its corresponding normal form.

We would like to a have a rewriting process on word that is terminating and confluent.



Normal form and word problem in Cactus groups

We would like to have a confluent and terminating rewriting process on words so that it provide the same final word for any two words that represent the same element.
One way to do this is that the rewriting process contains the defining relations with an orientation



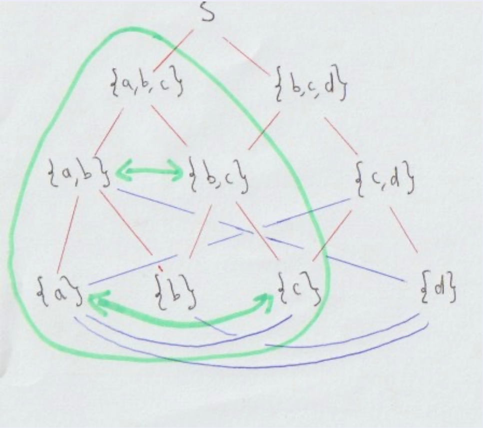
Example

in $\langle a, b \mid ab = ba \rangle$, the rewriting process $a^{\pm 1} b^{\pm 1} \to b^{\pm 1} a^{\pm 1}$, $a^{-1} a \to \varepsilon$, $b^{-1} b \to \varepsilon$, $aa^{-1} \to \varepsilon$, $bb^{-1} \to \varepsilon$ provide a normal form

Normal form and word problem in Cactus groups

Example

Consider the permutation groups S_5. This is a Coxeter group with Coxeter graph $a - b - c - d$ *and* $S = \{a, b, c, d\}$. *The set* P_S *has 10 elements :*



1. $\omega_{a,b,c}$ *permutes the* ω_X *with* X *in the green part.*
2. $\omega_{a,b}$ *permutes* ω_a *and* ω_b. *It commutes with* ω_d

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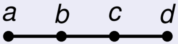
Consider the permutation groups S_5. This is a Coxeter group with Coxeter graph 

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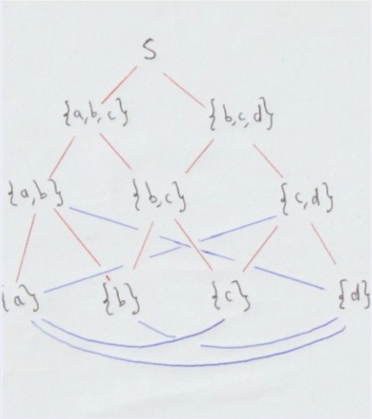

Do we get a normal form by pushing on the left using the total order ?

The answer is no : the process is not confluing.

$$\omega_a\omega_d\omega_{b,c,d} \to \omega_d\omega_a\omega_{b,c,d}$$

$$\omega_a\omega_d\omega_{b,c,d} \to \omega_a\omega_{b,c,d}\omega_b$$

The rewriting process is terminating but it is not confluent

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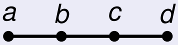
Consider the permutation groups S_5. This is a Coxeter group with Coxeter graph 

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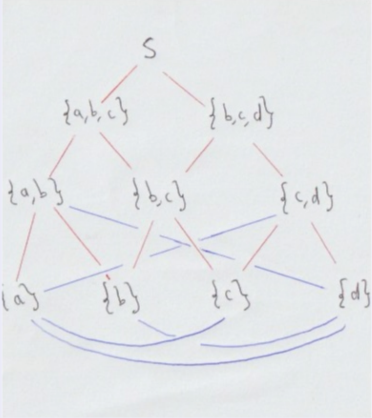

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$$\omega_a \omega_d \omega_{b,c,d} \to \omega_d \omega_a \omega_{b,c,d}$$

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Normal form and word problem in Cactus groups

Definition

1. A strata is a subset U of $P(S)$ so that for any two elements of U there is a relation $xy = ...$ (and so also a relation $yx = ...$)
2. to a stata $U = \{x_1, \cdots, x_k\}$ one can associate a unique word (a strata word) $x_1 \cdots x_k$ so that $i < j \Rightarrow x_i < x_j$ for the previous defined total order on $P(S)$.

Remark

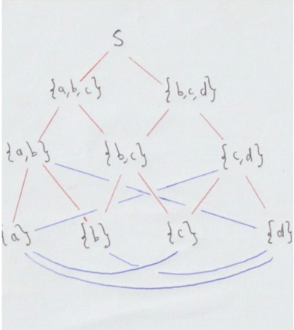
1. *A strata corresponds to the notion of a clique in a graph.*
2. *Any singleton of $P(S)$ is a strata, so any word can be decomposed as a product of strata word*

Example

In $J(S_5)$, $U_1 = \{\omega_a; \omega_{a,b}; \omega_d\}$ and $U_2 = \{\omega_S; \omega_{c,d}; \omega_c\}$ are stratas. Their associated stata words are $w(U_1) = \omega_{a,b}\omega_d\omega_a$ and $w(U_2) = \omega_S\omega_{c,d}\omega_c$.

Normal form and word problem in Cactus groups

Idea of the rewritting process : push/remove a letter from right to left from a strata to the previous one as long as possible (and remove empty strata). We want the process be compatible with strata words. We have to be careful !

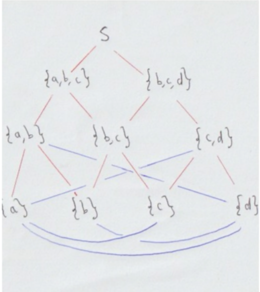


Let $w = \omega_{a,b}\omega_d\omega_a\omega_S\omega_{c,d}\omega_c = \omega_{a,b}\omega_d\omega_a \cdot \omega_S\omega_{c,d}\omega_c = w(U_1)w(U_2)$ with

$U_1 = \{\omega_a; \omega_{a,b}; \omega_d\}$ and $U_2 = \{\omega_S; \omega_{c,d}; \omega_c\}$.

Then $\{\omega_a; \omega_{a,b}; \omega_d; \omega_S\}$ and $\{\omega_S; \omega_{c,d}; \omega_c\}$ are strata, but $\omega_a\omega_{a,b}\omega_d\omega_S$ is not a strata word.

Normal form and word problem in Cactus groups



Let $w = \omega_{a,b}\omega_d\omega_a\omega_S\omega_{a,b}\omega_b = \omega_{a,b}\omega_d\omega_a \cdot \omega_S\omega_{a,b}\omega_b = w(U_1)w(U_2)$ with

$U_1 = \{\omega_a; \omega_{a,b}; \omega_d\}$ and $U_2 = \{\omega_S; \omega_{a,b}; \omega_b\}$.

We have $\omega_{a,b}\omega_d\omega_a\omega_S = \omega_S\omega_{c,d}\omega_d\omega_a$.

The latter is a strata word. The rewritting process is $U_1 \cdot U_2 \mapsto \{\omega_S; \omega_d; \omega_{c,d}; \omega_a\} \cdot \{\omega_{a,b}; \omega_b\}$

Then $\omega_{a,b}\omega_b = \omega_a\omega_{a,b}$ and

$$\omega_S\omega_{c,d}\omega_d\omega_a \cdot \omega_a\omega_{a,b} = \omega_S\omega_{c,d}\omega_d \cdot \omega_{a,b}$$

and the final rewritting process is :

$$\{\omega_S; \omega_d; \omega_{c,d}; \omega_a\} \cdot \{\omega_{a,b}; \omega_b\} \to \{\omega_S; \omega_{c,d}; \omega_d\} \cdot \{\omega_{a,b}\}$$

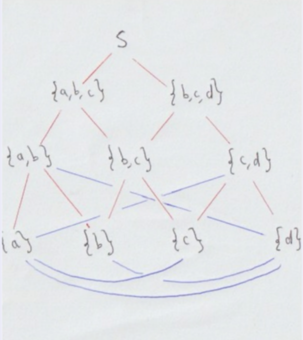
Normal form and word problem in Cactus groups

Proposition

The rewriting process is confluent and terminating : we get normal forms

Example

Consider the permutation groups S_5. : $\overset{a}{\bullet}\!\!-\!\!\!-\!\!\overset{b}{\bullet}\!\!-\!\!\!-\!\!\overset{c}{\bullet}\!\!-\!\!\!-\!\!\overset{d}{\bullet}$ *and* $S = \{a, b, c, d\}$.



$$\omega_a\omega_d\omega_{b,c,d} \to \omega_d\omega_a\omega_{b,c,d}$$

$$\omega_a\omega_d\omega_{b,c,d} \to \omega_a\omega_{b,c,d}\omega_b$$

$$\{\omega_a\}\{\omega_d\}\{\omega_{b,c,d}\} \to \{\omega_a;\omega_d\}\{\omega_{b,c,d}\}$$

$$\{\omega_a\}\{\omega_d\}\{\omega_{b,c,d}\} \to \{\omega_a\}\{\omega_{b,c,d};\omega_b\}$$

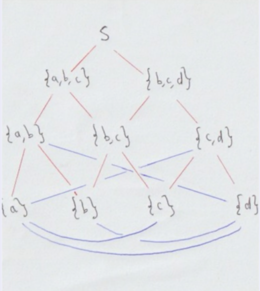
$$\to \{\omega_a;\omega_d\}\{\omega_{b,c,d}\}$$

Trickle groups

Proposition

What do we need to get this confluent and terminating rewriting process ?

Example



1. *a graph Γ with $P(S)$ for vertex set so that there is an edge between x and y when a relation $xy = \ldots$ exists.*
2. *a partial order $\leq$ on $P(S)$ so that if $x \leq y$ then there is an edge between x and y.*
3. *for x in $P(S)$, there is an automorphism $y \mapsto y^x$ of $\mathrm{star}_x(\Gamma)$ that fixes all the elements that are not smaller than x.*
4. *a map $n : P(S) \to \mathbb{N}_{\geq}2, x \mapsto n(x)$ that provides the order $n(x)$ of x.*
5. *some compatibility properties between these objects*

Trickle groups

Definition

A *trickle graph* is a tuple $(\Gamma, \leq, n, (\varphi_x)_{x \in V(\Gamma)})$, where

1. Γ is a simplicial graph
2. $\leqslant$ is a partial order on $V(\Gamma)$ so that (a) $x < y \Rightarrow \{x, y\} \in E(\Gamma)$;
3. $\varphi_x : \mathrm{star}_x(\Gamma) \to \mathrm{star}_x(\Gamma), y \mapsto y^x$ is a graph automorphism of $\mathrm{star}_x(\Gamma)$ such that (b) $z \leqslant y \iff \varphi_x(z) \leqslant \varphi_x(y)$ and (c) $\varphi_x(y) \neq y \Rightarrow y < x$
4. n is a labelling $n : V(\Gamma) \to \mathbb{N}_{\geqslant 2} \cup \{\infty\}$ so that (d) $n(\varphi_x(y)) = n(y)$ and (e) $o(\varphi_x)$ divides $n(x)$ (when the latter is finite).

with the two extra compatibilities rules :

- $\{x, y\} \in E_{||}(\Gamma)$ and $z \leqslant y \Rightarrow \{x, z\} \in E_{||}(\Gamma)$;
- $z < y < x \Rightarrow (z^y)^x = (z^x)^{y^x}$.

Trickle groups

Definition

Given a trickle graph $(\Gamma, \leq, n, (\varphi_x)_{x \in V(\Gamma)})$, the associated trickle group is defined by the following group presentation :

$$\mathrm{Tr}(\Gamma) = \left\langle V(\Gamma) \,\middle|\, \begin{array}{lcll} x^{n(x)} & = & 1 & ; x \in V, n(x) \neq \infty \\ \varphi_x(y)\, x & = & \varphi_y(x)\, y & ; \{x, y\} \in E(\Gamma) \end{array} \right\rangle$$

Example

1. *RAAG and RACG are trikle groups*
2. *Cactus groups associated with Coxeter groups are trickle groups*

Trickle groups

Proposition

If $(\Gamma, \leq, n, (\varphi_x)_{x\in V(\Gamma)})$ *is a trickle graph, there exists a terminating and confluent rewriting process on* $(V(\Gamma))^*$ *that provides a normal form* $\mathrm{Tr}(\Gamma) \to (V(\Gamma))^*$.

Proposition

If $(\Gamma, \leq, n, (\varphi_x)_{x\in V(\Gamma)})$ *then*

1. *The word problem is solvable in* $\mathrm{Tr}(\Gamma)$ *;*
2. *If* X *is a subset of* $V(\Gamma)$ *that is closed* φ_x *for* x *in* X *then the subgroup* $\mathrm{Tr}(\Gamma)_X$ *generated by* X *is a trickle group* $\mathrm{Tr}(\Gamma_X)$ *in the obvious way.*
3. $\mathrm{Tr}(\Gamma_X) \cap \mathrm{Tr}(\Gamma_Y) = \mathrm{Tr}(\Gamma_{X\cap Y})$
4. *In a cactus group a subgroup generated by a subset of not comparable elements is a RACG.*

Trickle groups

Proposition

If $(\Gamma, \leq, n, (\varphi_x)_{x \in V(\Gamma)})$ *is a trickle graph such that* $n(x) = \infty$ *for all* x *then*

1. *The submonoid* $\mathrm{Tr}^+(\Gamma)$ *of* $\mathrm{Tr}(\Gamma)$ *generated by* $V(\Gamma)$ *has the same presentation as* $\mathrm{Tr}(\Gamma)$ *(as a presentation of monoids).*
2. $\mathrm{Tr}^+(\Gamma) \cap \mathrm{Tr}(\Gamma_X) = \mathrm{Tr}^+(\Gamma_X)$
3. *The monoid* $\mathrm{Tr}^+(\Gamma)$ *is cancellative, possesses GCD and conditionnal LCM*
4. *The group* $\mathrm{Tr}(\Gamma)$ *is Garside (with Garside monoid* $\mathrm{Tr}^+(\Gamma)$*) if and only if* Γ *is complete.*

Dual cactus groups

Definition

Let (W,S) be a finite Coxeter system. By T denote its set of reflections. Fix a Coxeter element c.

Then (W,T,c) is a dual Coxeter system. An element δ of W is parabolic relatively to c if $\ell_T(w)+\ell_T(w^{-1}c)=\ell_T(c)$.

Example

in the symmetric group $\mathfrak{S}_{n+1}$, the element $s_1\cdots s_n$ is a Coxeter element.

Dual cactus groups

Proposition

Let $t_1 \cdots t_k = \delta$ be a decompostion over T of a parabolic element δ with $k = \ell_T(\delta)$. Then

1. *The subgroup W_δ of W generated by $t_1, \cdots, t_k$ depends on δ only.*
2. *$(W_\delta, T \cap W_\delta, \delta)$ is a dual Coxeter system.*
3. *$\delta = \delta' \iff W_\delta = W_{\delta'}$.*

There is a natural partial order on the set of parabolic elements relatively to c and a notion of irreducible parabolic elements.

Replacing the set of irreducible ω_X by the set of irreducible δ, we can define a notion of dual cactus groups in very similar way to the definition of cactus groups.

Dual cactus groups

Proposition

Dual cactus groups are trikle groups.

Example

Consider $W = < a, b \mid a^2 = b^2 = 1\,;\ aba = bab >$

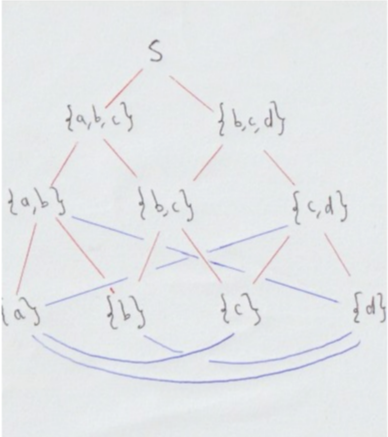
$$= < a, b, c \mid a^2 = b^2 = c^2 = 1\,;\ ab = bc = ca >.$$

Set $c = ab$. *The associated dual cactus groups has the following presentation :*

$$< a, b, c, d \mid a^2 = b^2 = c^2 = d^3 = 1; ad = dc; bd = da; cd = db >$$

Remark

1. *We also have examples of trickle groups associated with : right-ordered quandle and Virtual braid groups*
2. *The Thompson group is a trickle group.*



Thanks

Quandle

Definition

1. a *quandle* is a pair $(Q, \leq)$ with Q a set and $\star : Q \times Q \to Q$ so that
   (1) $\forall x \in Q, x \star x = x$;
   (2) $\forall x, y, z$ in $Q, (x \star z) \star (y \star z) = (x \star y) \star z$;
   (3) For all $x \in Q$, l $y \mapsto y \star x$ is one-to-one on Q.
2. A quandle is right-ordered when there is a total order on Q such that $y \leq z \Rightarrow y \star x \leq z \star x$.

Example

Take $\mathbb{R}$ with its natural total order. Fix $\alpha > 1$ in $\mathbb{R}$ and set $x \star y = \alpha x + (1 - \alpha) y$. Then $(\mathbb{R}, \star)$ is a right-ordered quandle.

Proposition

Let $(Q, \star, \leq)$ be a right-ordered quandle. The following group is trickle

$$\mathrm{Tr}(\mathrm{Q}) = \langle \mathrm{c_x}, \mathrm{x} \in \mathrm{Q} \mid \mathrm{c_{y \star x} c_x} = \mathrm{c_x c_y} \; \mathit{for} \; \mathrm{y} < \mathrm{x} \rangle$$