Garside groups and the Yang-Baxter equation

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"Set-theoretical" solutions of the quantum Yang-Baxter equation and a class of Garside groups

To appear in Communications in Algebra

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The quantum Yang-Baxter equation - QYBE

Let $R : V \otimes V \to V \otimes V$ be a linear operator, where V is a vector space. The QYBE is the equality $R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12}$ of linear transformations on $V \otimes V \otimes V$, where R^{ij} means R acting on the $i-$th and $j-$th components.

A set-theoretical solution (X, S) of this equation [Drinfeld]

- V is a vector space spanned by a set X.
- R is the linear operator induced by a mapping $S : X \times X \to X \times X$.

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$S(i, j) = (g_i(j), f_j(i))$, where $f_i, g_i : X \to X$.

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Our object of study: The structure group of (X, S)

Assumption: (X, S) is a non-degenerate, involutive and braided solution.

The structure group G of (X, S) [Etingof, Schedler, Soloviev]

- The generators: the elements of X
- The defining relations: $x_i x_j = x_k x_l$ whenever $S(i, j) = (k, l)$

There are exactly $\frac{n(n-1)}{2}$ relations.

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The example

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$.

The functions that define S

Let $f_1 = g_1 = (1, 2, 3, 4)(5)$
$f_2 = g_2 = (1, 4, 3, 2)(5)$
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$f_4 = g_4 = (1, 4, 3, 2)(5)$
$f_5 = g_5 = (1)(2)(3)(4)(5)$

(X, S) is a non-degenerate, involutive and braided solution.

The defining relations in G and in M (the monoid with the same pres.)

$$\begin{array}{ll} x_1^2 = x_2^2 & x_3^2 = x_4^2 \\ x_1x_2 = x_3x_4 & x_1x_5 = x_5x_1 \\ x_1x_3 = x_4x_2 & x_2x_5 = x_5x_2 \\ x_2x_4 = x_3x_1 & x_3x_5 = x_5x_3 \\ x_2x_1 = x_4x_3 & x_4x_5 = x_5x_4 \end{array}$$

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x_2x_1 = x_4x_3 & x_4x_5 = x_5x_4
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Our results: Theorem A

Theorem

Let (X, S) be a non-degenerate, involutive and braided set-theoretical solution of the quantum Yang-Baxter equation with structure group G. Then G is Garside.

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Theorem

Let (X, S) be a non-degenerate, involutive and braided set-theoretical solution of the quantum Yang-Baxter equation with structure group G. Then G is Garside.

Sketch of the proof: Recognizing Garside monoids [P.Dehornoy 2002]

A monoid M is Garside if and only if

- M is atomic.
- M is right cancellative.
- M satisfies the right cube condition on the set of atoms.
- M has a finite generating set S closed under complement, that is if $U, V \in S$ then the complement $U \setminus V$ is in S.

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Theorem

Let (X,S) *be a non-degenerate, involutive and braided set-theoretical solution of the quantum Yang-Baxter equation with structure group* G*. Then* G *is Garside.*

Sketch of the proof

- Expressing $x_i \setminus x_j$ in terms of the functions g_i:
  Let x_i, x_j be different elements in X.
  Then $x_i \setminus x_j = g_i^{-1}(j)$.

In a dual way, $x_i \widetilde{\setminus} x_j = f_j^{-1}(i)$.

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- The right cube condition is satisfied on X.

Theorem A: The converse

Assume that $\mathrm{Mon}\langle X \mid R\rangle$ is a **Garside monoid** such that:
- the cardinality of R is $n(n-1)/2$, where n is the cardinality of X.
- each side of a relation in R has length 2.
- if the word $x_i x_j$ appears in R, then it appears only once.

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Then there exists a function $S : X \times X \to X \times X$ such that

- (X, S) is a non-degenerate, involutive and braided set-theoretical solution.
- $G = \mathrm{Gp}\langle X \mid R\rangle$ is its structure group.

Sketch of the proof

Definition of (X,S) non-degenerate and involutive

- For each relation $x_i x_j = x_k x_l$ in R, define $S(x_i, x_j) = (x_k, x_l)$.
- (X,S) is non-degenerate, since $\mathrm{Mon}\langle X \mid R\rangle$ is Garside.
- (X,S) is involutive, since each $x_i x_j$ in R appears only once.

Proof of (X,S) braided

Assume (X,S) is non-degenerate and involutive.
Then (X,S) is braided if and only if the right cube condition and the left cube condition are satisfied on X.

(X,S) is braided $\Leftrightarrow g_i g_j = g_{g_i(j)} g_{f_j(i)}$ and $f_j f_i = f_{f_j(i)} f_{g_i(j)}$
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Our results: Theorem B

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- *The right lcm of the generators is a Garside element.*
- *The Garside element has length n.*
- *The (co)homological dimension of the structure group G is n. [P.Dehornoy, Y.Laffont 2003] [R.Charney, J.Meier, K.Whittlesey 2004] [J. McCammond]*

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Sketch of the proof: Characterization of the simples (1)

Who are the simples?

- A simple element s is a right lcm of some subset of generators Y.
- The set of simples χ is equal to $\overline{X}^{\vee} \cup \{1\}$.

Proof of $\chi = \overline{X}^{\vee} \cup \{1\}$

- $\chi = \overline{X}^{\backslash,\vee}$ [P.Dehornoy 2002]
- Induction on the number of steps in the construction of χ:
  - $M \setminus x \subseteq X \cup \{1\}$.
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What is the length of a simple?

- The length of s is equal to $|Y|$.
- The length of Δ is equal to $|X|$.

Proof

- Induction on $|Y|$
  - The length of s is less or equal than $|Y|$.
  - $(\vee_i x_i) \setminus x_j \neq 1$ (all the x_i, x_j are dft).

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Let (X,S) be a non-degenerate, involutive and braided set-theoretical solution of the QYBE.

Definition

(X,S) is *decomposable* if it is the union of two nonempty disjoint non-degenerate invariant subsets. Otherwise, (X,S) is *indecomposable*.

Theorem (Etingof,Schedler,Soloviev)

(X,S) is indecomposable if and only if G acts transitively on X, where $x_i \to g_i^{-1}$ is a right action of G on X.

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Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and S as before.

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- $X = \{x_1, x_2, x_3, x_4\} \cup \{x_5\}$.
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The defining relations in G and in M

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\begin{array}{lll}
x_1^2 = x_2^2 & x_3^2 = x_4^2 & (x_5 x_5 = x_5 x_5) \\
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Our results: Theorem C

Theorem

Let (X, S) be a non-degenerate, involutive and braided set-theoretical solution of the quantum Yang-Baxter equation with structure group G. Then
(X, S) is indecomposable if and only if G is $\Delta-$pure Garside.

A consequence

If (X, S) is indecomposable then the center of G is cyclic, generated by some exponent of Δ.

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$\Delta-$pure Garside monoids [Picantin 2001]

Definition of a $\Delta-$pure Garside monoid

Let M be a Garside monoid. Then M is $\Delta-$*pure* if for every x, y in X, it holds that $\Delta_x = \Delta_y$,
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If M is a $\Delta-$pure Garside monoid, Δ is its Garside element and G its group of fractions. Then the center of M (resp. of G) is the infinite cyclic submonoid (resp. subgroup) generated by Δ^e, where e is a natural number.

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Sketch of the proof of Theorem C

A correspondence between $w \setminus x$ and $g_w^{-1}(x)$

If $w \setminus x \neq 1$, then $w \setminus x = g_w^{-1}(x)$:

$$
\begin{tikzcd}
{} \arrow[r, "h_1"] \arrow[d, "x"'] & {} \arrow[r, "h_2"] \arrow[d, "g_1^{-1}(x)"] & {} \arrow[r, dashed, no head] \arrow[d, "g_2^{-1}g_1^{-1}(x)"] & {} \arrow[r, "h_k"] & {} \arrow[d, "g_k^{-1}\cdots g_2^{-1}g_1^{-1}(x)"] \\
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Non-involutive permutation solutions are Garside (1)

Permutation solutions [Lyubashenko]

A permutation solution: $S(x, y) = (g(y), f(x))$, where $f, g : X \to X$.

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An equivalence relation on the set X

$x \equiv x'$ if and only if there is an integer k such that $(fg)^k(x) = x'$.

A new permutation solution

- $X' = X/\equiv$
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- $G' = \mathrm{Gp}\langle [x_1], [x_2], [x_5] \mid [x_1]^2 = [x_2]^2, [x_1][x_5] = [x_5][x_2], [x_2][x_5] = [x_5][x_1]\rangle$.

Note that G and G' have the same presentation, up to a renaming of the generators.

Example

A non-involutive permutation solution (X, S)

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and let $f = (1,4)(2,3)$ and $g = (1,2)(3,4)$:

$$\begin{array}{ll} x_1^2 = x_2x_4 = x_3^2 = x_4x_2 & x_1x_2 = x_1x_4 = x_3x_4 = x_3x_2 \\ x_2^2 = x_1x_3 = x_4^2 = x_3x_1 & x_1x_5 = x_5x_4 = x_3x_5 = x_5x_2 \\ x_2x_1 = x_2x_3 = x_4x_3 = x_4x_1 & x_2x_5 = x_5x_3 = x_4x_5 = x_5x_1 \end{array}$$

$G = \mathrm{Gp}\langle x_1, x_2, x_5 \mid x_1^2 = x_2^2, x_1x_5 = x_5x_2, x_2x_5 = x_5x_1\rangle$.

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Non-involutive permutation solutions are Garside (3)

Sketch of the proof

- (X', S') is a well-defined non-degenerate, involutive and braided solution (a permutation solution).
- The structure group of (X', S') is isomorphic to G.

The relation $\equiv$ imitates the cancellation

If $x \equiv x'$, then x and x' are equal in G.

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Thank you!!

Recognizing Garside monoids

A criteria for recognizing Garside monoids [P.Dehornoy 2002]

A monoid M is Garside if and only if

- M is atomic.
- M is right cancellative.
- M satisfies the right cube condition on the set of atoms.
- M has a finite generating set S closed under complement, that is if $U, V \in S$ then the complement $U \setminus V$ is in S.

Proposition [P.Dehornoy 2002]

If M (atomic) satisfies the right cube condition on the set of atoms then

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