Atelier autour des groupes de R. Thompson

The geometry of the Cayley graph

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Introduction

All along this talk we will be interested in Thompson's group F, not T or V

Its presentation is:

$$F = \left\langle x_0, x_1 \mid [x_0x_1^{-1}, x_2], [x_0x_1^{-1}, x_3] \right\rangle$$

Every time we use a notion which involves finite presentation, such as Cayley graph, word metric, it will always be with this presentation

Observe though that the two relators

$$x_0x_1^{-1}x_0^{-1}x_1x_0x_1x_0^{-1}x_0^{-1}x_1^{-1}x_0$$

$$x_0x_1^{-1}x_0^{-2}x_1x_0^2x_1x_0^{-1}x_0^{-2}x_1^{-1}x_0^2$$

are awkward to work with. Also, since these relators do not have a nice form, the shape of the Cayley graph is difficult to grasp.

So, to understand elements we will use the infinite presentation and the realization of the group as pairs of binary trees (John Meier's deuxième présentation)

The infinite presentation

$$\left\langle x_i,\ i \geq 0 \mid x_j x_i = x_i x_{j+1},\ i < j \right\rangle$$

is much better because of its regularity and symmetry.

Recall that

$$x_n = x_0^{-n+1} x_1 x_0^{n-1}$$

The normal form

An element written in any way in the infinite presentation can be reduced to its normal form.

The relators

$$x_j x_i = x_i x_{j+1}$$

admit several different rewritings: the expressions

$$x_i^{-1} x_j = x_{j+1} x_i^{-1}$$
$$x_j^{-1} x_i = x_i x_{j+1}^{-1}$$

allow to move all generators with positive exponents to the left, and with negative ones to the right.

And the expressions

$$x_j x_i = x_i x_{j+1}$$
$$x_i^{-1} x_j^{-1} = x_{j+1}^{-1} x_i^{-1}$$

allow to order the indices, increasingly in the positive part and decreasingly in the negative one.

So every element admits a normal form of the type

$$x_0^{b_0} x_1^{b_1} \dots x_n^{b_n} x_m^{-a_m} \dots x_1^{-a_1} x_0^{-a_0}$$

with

$$a_i \geq 0 \qquad b_j \geq 0$$

Uniqueness of the normal form

This normal form is unique, provided it satisfies an extra condition.

Theorem:

- Every element of F admits an expression of the form

$$x_0^{b_0} x_1^{b_1} \dots x_n^{b_n} x_m^{-a_m} \dots x_1^{-a_1} x_0^{-a_0}$$

- For each element of F, there is exactly one such form which satisfies the following condition: if $a_i \neq 0$ and $b_i \neq 0$ simultaneously, then either $a_{i+1} \neq 0$ or $b_{i+1} \neq 0$.

For a proof of this result, check [Brown–Geoghegan]

The idea is that the shape of the relators allows for congujation to a smaller normal form if the condition is not satisfied: for instance, the element

$$x_0 x_1^3 x_3 x_5 x_7^{-1} x_6^{-4} x_3^{-2} x_2^{-1}$$

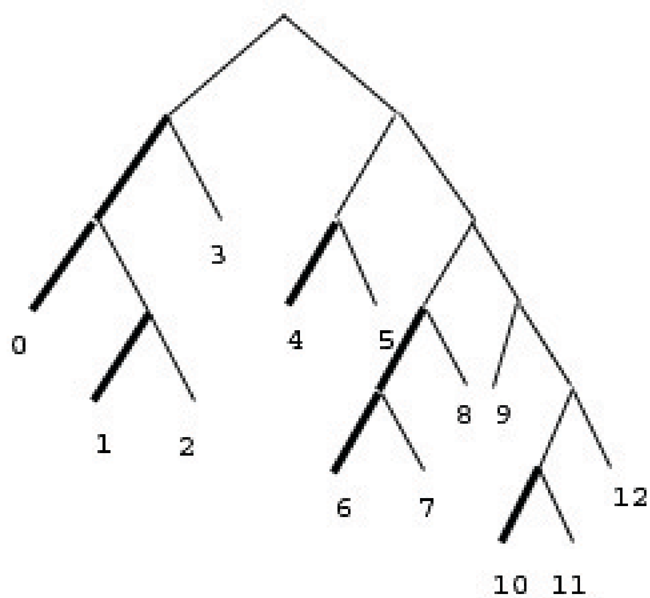
can be reduced to

$$x_0 x_1^3 x_4 x_6^{-1} x_5^{-4} x_3^{-1} x_2^{-1}$$

applying the relator several times. This last expression is the normal form.

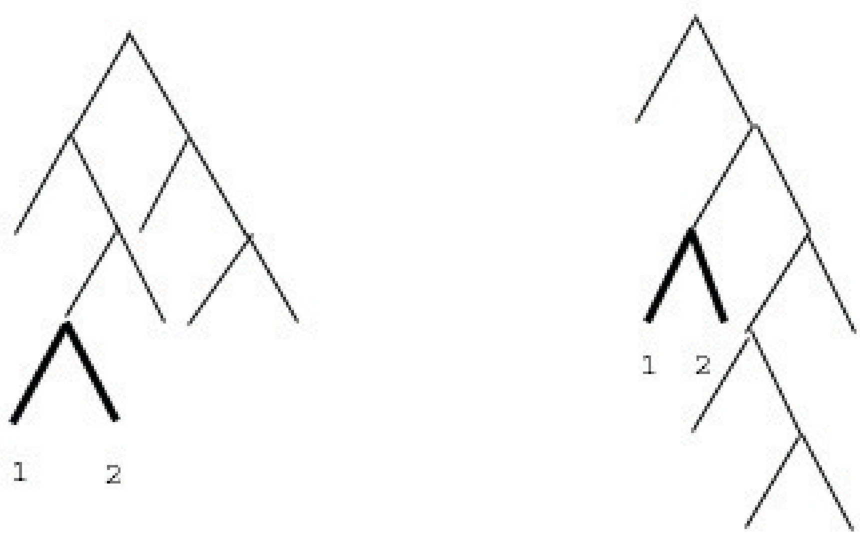
Normal form vs. tree diagram

As seen in John's talk, normal forms and diagrams are in 1-to-1 correspondence:



$$x_0^2 x_1 x_4 x_6^2 x_{10}$$

It is interesting to observe that nonreduced diagrams correspond to reducible normal forms. Observe that the fact that $a_i \neq 0$ and $b_i \neq 0$ means that we have both leaves labelled i with their carets, and if $a_{i+1} = 0$ and $b_{i+1} = 0$ then that means exactly that the two carets are exposed.



$$x_0\mathbf{x_1^2}x_4^{-1}x_3^{-1}\mathbf{x_1^{-1}} = x_0x_1x_3^{-1}x_2^{-1}$$

When the carets are reduced, all indices larger than them get reduced by one, exactly as in the normal form.

Hence, under the correspondence given above, unique normal forms correspond to reduced diagrams and viceversa.

Distances in the Cayley graph

Even though they are given in terms of the infinite presentation, normal forms are useful to give values for the distance in the Cayley graph (which relates only to x_0 and x_1).

Recall that the norm $|x|$ of an element x is the distance from this element to the identity, i.e. the length of the shortest expression that can be written for this element in terms of the generators x_0 and x_1. Equivalently, the length of the shortest path in the Cayley graph from the identity to x.

An interesting result is the following:

Theorem. Let $x \in F$ have normal form

$$x_0^{b_0} x_1^{b_1} \dots x_n^{b_n} x_m^{-a_m} \dots x_1^{-a_1} x_0^{-a_0},$$

and consider the following number:

$$D(x) = a_0 + a_1 + \dots + a_m + b_0 + b_1 + \dots + b_n + m + n.$$

Then, there exists a constant C such that, for every x, we have

$$\frac{D(x)}{C} \leq |x| \leq C\, D(x).$$

Proof: The upper bound is easy: given the normal form, replace each term x_i by its value $x_0^{-i+1}x_1x_0^{i-1}$, and after a little cancellation one can see that the word obtained in terms of x_0 and x_1 has length less than $3D(x)$.

For the lower bound, start with the shortest word in x_0 and x_1 which represents x, and follow the procedure to find the normal form for x from this expression. Clearly, the normal form will have a shorter length, due to cancellations or to the normal form reduction. So we have

$$a_0 + a_1 + \ldots + a_m + b_0 + b_1 + \ldots + b_n \leq |x|.$$

And notice as well that the highest exponent that can appear in the normal form comes from a generator x_1 which is moved around, increasing its index in 1 for each move. It can only be moved at most $|x| - 1$ times, so its subindex will end up being at most equal to $|x|$. From this we get:

$$n \leq |x| \qquad m \leq |x|.$$

And combining all three inequalities we get

$$\frac{D(x)}{3} \leq |x|,$$

to finish the proof. $\square$

This result has interesting corollaries.

Recall that the translation number of an element x in a finitely generated group G is the number

$$tr(x) = \lim_{k\to\infty} \frac{|x^k|}{k}.$$

This number is zero when the subgroup $\langle x \rangle$ is distorted in G, and nonzero when the cyclic subgroup is not distorted.

Corollary. All translation numbers in F are strictly positive. Hence, all cyclic subgroups are undistorted.

Proof. Let x have the usual normal form

$$x_0^{b_0} x_1^{b_1} \ldots x_n^{b_n} x_m^{-a_m} \ldots x_1^{-a_1} x_0^{-a_0},$$

and let i_0 be the smallest index appearing in the normal form, i.e. $a_i = b_i = 0$ if $i < i_0$. Assume, by taking inverses if necessary, that $b_{i_0} \neq 0$, and since the translation number is invariant under conjugation, we can also assume that $a_{i_0} = 0$. Then, the first term of the normal form for x^k must be $x_{i_0}^{kb_{i_0}}$. From here:

$$tr(x) = \lim_{k\to\infty} \frac{|x^k|}{k} \geq \lim_{k\to\infty} \frac{D(x^k)}{3k} \geq \frac{b_0}{3} > 0,$$

which completes the proof. □

Other corollaries:

Corollary. The subgroups of F isomorphic to $F \times F$ and $F \times \mathbb{Z}$ are quasi-isometrically embedded.

Corollary. The abelian subgroup of rank n generated by

$$x_0 x_1^{-1} \quad x_2 x_3^{1} \quad \cdots \quad x_{2n-2} x_{2n-1}^{-1}$$

is quasi-isometrically embedded

Proof. Exercise.

Corollary. F has exponential growth.

Proof. It is enough to prove that the set of elements x which have

$$D(x) \leq n$$

is exponential in n. But this reduces to a counting combinatorial problem. For instance, the number of elements of the type

$$x_0^{b_0} x_1^{b_1} \dots x_n^{b_n}$$

with $0 \leq b_i \leq n$ for all $i = 0, 1, \dots, n$ and satisfying

$$\sum_i b_i = n$$

is of the order of

$$\binom{2n}{n} \sim 4^n$$

Number of carets

Given an element $x \in F$, let $N(x)$ be the number of carets of either one of the trees appearing in the *reduced* tree diagram for x. The number $N(x)$ is also a good indication of the norm of the element x.

Theorem. There exists a constant C such that, for every element $x \in F$, we have

$$\frac{N(x)}{C} \leq |x| \leq C\,N(x)$$

Proof. As usual, the upper bound is easy to prove. From the relation between the normal form and the tree diagram, it is not diificult to see that $N(x) \leq D(x) + 1$, from which the result is easily deduced. In any case, the exact number of carets in terms of the numbers a_i and b_j is computed exactly in [Cleary-Taback].

For the lower bound, observe that since the generators x_0 and x_1 have exactly three and four carets, and that by multiplying two elements the number of carets of the product is at most the sum of the number of carets, we have that

$$N(x) \leq 4|x|$$

from which the lower bound follows immediately. $\square$

The exact value of the norm

Fordham's method

In 1995, in his Ph.D. Thesis, S. Blake Fordham studied many properties of Thompson's group F. The main result of the thesis is an algorithm which allows the computation of the *exact* value of the norm of an element knowing only the reduced tree diagram (or, equivalently, the unique normal form).

Fordham's method consists in labelling each caret with one of seven given types. Then, each caret in the first tree is paired with its corresponding caret of the second tree, preserving the order. And finally, to each pair of carets, a weight is assigned according to the labels of the two carets, and given by a double-entry table.

Types of carets

A caret is called right or left caret if it appears in the right or left edge of the binary tree. Equivalently, a caret is left if the interval it represents in the subdivision tree of $[0,1]$ contains 0, and right if it contains 1. A caret which is not right or left is called interior. The root caret is counted as left for the purposes of this computation.

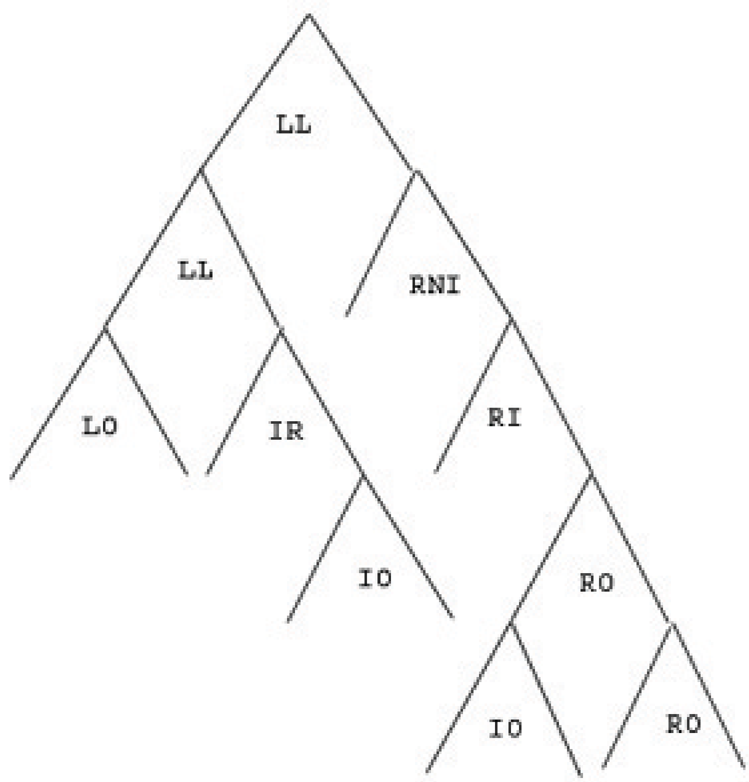
The carets immediately below a given caret are called its right and left children.

According to Fordham's method, the carets in a tree are assigned one of the following seven types:

- Type $\mathcal{L}_\varnothing$. A left caret which has not a left child. There is always one and only one caret of this type.
- Type $\mathcal{L}_L$. A left caret which has a left child.
- Type $\mathcal{R}_\varnothing$. A right caret such that all its descendants (i.e. carets below it) on the right side are right (and hence also of type $\mathcal{R}_0$).
- Type $\mathcal{R}_{\mathrm{i}}$. A right caret such that its right child has a left child.
- Type $\mathcal{R}_{\mathrm{ni}}$. A right caret such that its right child has no left child, but one of its further descendants has a left child.

- Type $\mathcal{I}_\emptyset$. An interior caret which has no right child.

- Type $\mathcal{I}_R$. An interior caret which has a right child.



Example

Fordham's table

	$\mathcal{R}_\varnothing$	$\mathcal{R}_{\mathsf{ni}}$	$\mathcal{R}_{\mathsf{i}}$	$\mathcal{L}_L$	$\mathcal{I}_\varnothing$	$\mathcal{I}_R$
$\mathcal{R}_\varnothing$	0	2	2	1	1	3
$\mathcal{R}_{\mathsf{ni}}$	2	2	2	1	1	3
$\mathcal{R}_{\mathsf{i}}$	2	2	2	1	3	3
$\mathcal{L}_L$	1	1	1	2	2	2
$\mathcal{I}_\varnothing$	1	1	3	2	2	4
$\mathcal{I}_R$	3	3	3	2	4	4

Theorem. The total weight assigned to an element $x \in F$ given by the values on Fordham's table on the pairs of carets of the reduced tree diagram coincides exactly with the norm of x with respect to the generators x_0 and x_1.

The proof is developed in Fordham's thesis and in the paper [Fordham] that resulted from it. The proof is long and very technical.

As a corollary, we can recover the theorem above which states that the value of the norm of an element x is equivalent to the number of carets in each tree of the reduced tree diagram. Since each caret pair is given a value between 0 and 4, and the number of carets with 0 is very limited, the result follows easily.

References

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